THE DENSITY FUNCTION OF THE FIRST OCCURRENCE OF A BINARY PATTERN

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Abstract

The probability density function associated with the first occurrence of a binary pattern in a sequence of independent Bernoulli trials was given in McDermott and Sheahan. We give a simpler expression for it in terms of a unique family of homogeneous polynomials in two variables. In the process, we obtain a recurrence relation for the density function and we study the coefficients of the homogeneous polynomials.

1. Introduction

Consider repeated independent Bernoulli trials in each of which the probability of *success* S is p with $0 < p < 1$, and so the probability of *failure* F is $q = 1 - p$. Fix a pattern π consisting of a Ss and b Fs where $a + b > 0$. We say that the pattern π *first occurs* on the nth of a sequence of independent Bernoulli trials if the sequence of outcomes for trials $n - a - b + 1$ to n forms the pattern π and no sequence of outcomes for trials i to $i + a + b - 1$ with $i \leq n - a - b$ forms this pattern. For example, the pattern SFS first occurs in the sequence of outcomes $SSFFSFSFS$ on trial 7. (Our definition differs from that of [4] whose authors would say that SFS first occurs on trial 5, that is, they count the trial on which the first occurrence begins.)

Let X_π be the random variable that counts the number of independent Bernoulli trials until the first occurrence of π and let f_π denote the probability density function of X_π. In [4], the authors found an expression for $f_\pi(n)$, $n \geq a+b$. One objective of the present work is to find a simpler expression. Specifically, we will produce, for each $n \geq 0$, a unique (in the sense of Proposition 1 below) homogeneous polynomial $P_n(x, y) = \sum_{i=0}^{n} c(n, i)x^{n-i}y^i \in \mathbb{Z}[x, y]$ of degree n with nonnegative integer coefficients such that

$$f_\pi(n + a + b) = p^a q^b P_n(p, q).$$

We obtain formulas for the coefficients $c(n, i)$. Along the way, and more interestingly,

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we establish recurrence relations of order $a+b$ for f_π and the $c(n,i)$, and we find the generating function for the $c(n,i)$.

2. Notation

As above, we assume the pattern π consists of a Ss and b Fs with length $a+b>0$. Following Imling [3], we say that a pattern μ of length at least one is an *initial pattern* of π if μ occurs at the beginning of π; a *final pattern* of π if it occurs at the end of π; an *initial-final pattern* of π if $\mu \neq \pi$ and μ is both an initial and final pattern of π, and a *maximal initial-final pattern* of π if it is an initial-final pattern of π that is longer than any other initial-final pattern of π. It was [3] that first drew our attention to the importance of the notion of initial-final pattern.

For example, $\pi_1 = SSS$ is a maximal initial-final pattern of $\pi = SSSS$, $\pi_2 = SS$ is a maximal initial-final pattern of π_1, $\pi_3 = S$ is a maximal initial-final pattern of π_2, and π_3 has no initial-final pattern. If $\mu = SFFSFSFSFFSF$, then $\mu_1 = SFFSF$ is a maximal initial-final pattern of μ, with $\mu_2 = SF$ a maximal initial-final pattern of μ_1. It is clear that a pattern either has a unique maximal initial-final pattern or no initial-final pattern. In the latter case we say, following [4], that it is *aperiodic*. The term *bifix-free* is also used.

We fix the following notation for the rest of this paper. Let $\pi_0 = \pi$. Suppose that, if it exists, π_{i+1} is the maximal initial-final pattern of π_i for $0 \le i \le k-1$ and that π_k is aperiodic. For example, if $\pi = \pi_0 = SFSFS$, then $\pi_1 = SFS$, $\pi_2 = S$, and $k=2$. Suppose also that the pattern π_i consists of a_i Ss and b_i Fs for $0 \le i \le k$. So $a = a_0 \ge a_1 \ge a_2 \ge \ldots \ge a_k \ge 0$, $b = b_0 \ge b_1 \ge b_2 \ge \ldots \ge b_k \ge 0$, and $a_0+b_0 > a_1+b_1 > \ldots > a_k+b_k > 0$. When $1 \le i \le k$, denote by μ_i the pattern such that $\pi = \pi_i\mu_i$. For example, if $\pi = SFSFS$, $\mu_1 = FS$ and $\mu_2 = FSFS$. For $0 \le i \le k$, define $c_i = a_0 - a_i$ and $d_i = b_0 - b_i$. Thus μ_i consists of c_i Ss and d_i Fs.

3. The polynomial P_n

The polynomial P_n, introduced in Section 1, is very easy to identify. For $n \ge 0$, consider the set T_n of sequences of trial outcomes of length $n+a+b$ ending in a first occurrence of the pattern π on trial $n+a+b$. Define $c(n,i)$ as the number of sequences in T_n that consist of $(a+n-i)$ Ss and $(b+i)$ Fs. Then if we define a polynomial $P_n(x,y)$ by the formula

$$P_n(x,y) = \sum_{i=0}^{n} c(n,i)x^{n-i}y^i,$$

we see that this polynomial is clearly homogeneous of degree n, has nonnegative integer coefficients, and it is easy to see that it satisfies

$$f_\pi(n+a+b) = p^a(1-p)^b P_n(p,1-p)$$

for every $p \in (0,1)$.

To find a formula for $c(n,i)$, we will provide an alternative description of these coefficients and then apply the following uniqueness result.

Proposition 1. *The polynomial P_n is unique in the following sense: if Q_n is a homogeneous polynomial in two variables of degree n such that $P_n(p, 1-p) = Q_n(p, 1-p)$ for every $p \in (0,1)$, then $P_n = Q_n$.*

Proof. I am grateful to my colleague Caryn Werner for providing the idea for the following proof.

Define the polynomial R_n by $R_n(z) = P_n(z, 1)$ and the polynomial S_n by $S_n(z) = Q_n(z, 1)$. Each of R_n and S_n is a polynomial in one variable of degree n. Now

$$\begin{aligned} &(1-p)^n(R_n(p/(1-p)) - S_n(p/(1-p))) \\ &= (1-p)^n(P_n(p/(1-p), 1) - Q_n(p/(1-p), 1)) \\ &= P_n(p, 1-p) - Q_n(p, 1-p) \\ &= 0 \end{aligned}$$

for every $p \in (0,1)$. Thus the polynomial $R_n - S_n$ has infinitely many roots and so is the zero polynomial. But P_n and R_n have the same coefficients, as do Q_n and S_n. Thus $P_n = Q_n$. ■

4. Two recurrence relations for f_π

Proposition 2. *The probability density function f_π satisfies $f_\pi(0) = f_\pi(1) = \ldots = f_\pi(a+b-1) = 0$, $f_\pi(a+b) = p^a q^b$, and*

$$\begin{aligned} f_\pi(n) = f_\pi(n-1) + &\sum_{j=1}^{k}(f_\pi(n-1-c_j-d_j) - f_\pi(n-c_j-d_j))p^{c_j}q^{d_j} \\ &-f_\pi(n-a-b)p^a q^b \end{aligned}$$

for $n \geq a+b+1$.

Proof. One source for the idea of this proof is [2].

Suppose that $n \geq a+b+1$. Consider the event E consisting of all possible sequences of outcomes of n trials where the last $a+b$ outcomes form π and where π does not occur on trials 1 through $n-a-b$. Note that elements of E may have an appearance of π before trial n also, as long as such an occurrence is between trials $n-a-b+1$ and $n-1$. The probability of E is

$$(1 - f(1) - \ldots - f(n-a-b))p^a q^b.$$

Now E is the union of $k+1$ disjoint events: E_0, where the first occurrence of π occurs on trial n; and E_j with $1 \leq j \leq k$, where the first occurrence of π is on trial $n - c_j - d_j$ and is followed by the pattern μ_j. The probability of E_0 is $f_\pi(n)$, whereas the probability of E_i is $f_\pi(n-c_j-d_j)p^{c_j}q^{d_j}$. Thus

$$(1-f_\pi(1)-\ldots-f_\pi(n-a-b))p^aq^b = f_\pi(n)+\sum_{j=1}^{k} f_\pi(n-c_j-d_j)p^{c_j}q^{d_j}.$$

So

$$f_\pi(n) = (1-f_\pi(1)-\ldots-f_\pi(n-a-b))p^aq^b - \sum_{j=1}^{k} f_\pi(n-c_j-d_j)p^{c_j}q^{d_j}.$$

Now

$$f_\pi(n-1) = (1-f_\pi(1)-\ldots-f_\pi(n-1-a-b))p^aq^b - \sum_{j=1}^{k} f_\pi(n-1-c_j-d_j)p^{c_j}q^{d_j}.$$

Subtracting yields

$$\begin{aligned} f_\pi(n) = f_\pi(n-1) &+ \sum_{j=1}^{k}(f_\pi(n-1-c_j-d_j)-f_\pi(n-c_j-d_j))p^{c_j}q^{d_j} \\ &- f_\pi(n-a-b)p^aq^b. \end{aligned}$$

■

Using the recurrence relation of Proposition 2 to compute $f_\pi(n)$ for $n \geq a+b$ will not result in the expression of $f_\pi(n)$ as a homogeneous polynomial of degree n in p and q. To get such an expression for $f_\pi(n)$ we must first 'homogenise' the recurrence relation as

$$\begin{aligned} f_\pi(n) = f_\pi(n-1)(p+q) &+ \sum_{j=1}^{k} f_\pi(n-1-c_j-d_j)(p+q)p^{c_j}q^{d_j} \\ &- \sum_{j=1}^{k} f_\pi(n-c_j-d_j)p^{c_j}q^{d_j} - f_\pi(n-a-b)p^aq^b. \quad \text{(HRR)} \end{aligned}$$

Using this relation to compute $f_\pi(n)$ for $n \geq a+b$ will result in the expression of $f_\pi(n)$ as a homogeneous polynomial of degree n in p and q, which by Proposition 1 must be $p^aq^bP_n(p,q)$.

Example 1. Let $\pi = SFS$. Then $a_0 = 2$, $b_0 = 1$, $a_1 = 1$, and $b_1 = 0$. Therefore $c_1 = 1$ and $d_1 = 1$. The relation (HRR) specialises to

$$\begin{aligned} f_\pi(n) &= f_\pi(n-1)(p+q) + f_\pi(n-3)(p+q)pq - f_\pi(n-2)pq - f_\pi(n-3)p^2q \\ &= f_\pi(n-1)(p+q) - f_\pi(n-2)pq + f_\pi(n-3)pq^2. \end{aligned}$$

Then $f_\pi(3) = p^2q$, $f_\pi(4) = p^3q + p^2q^2$, $f_\pi(5) = p^4q + p^3q^2 + p^2q^3$, and $f_\pi(6) = p^5q + p^4q^2 + 2p^3q^3 + p^2q^4$.

One can use (HRR) to calculate the 'homogenised' probability generating function for X_π as

$$G_\pi(s) = \frac{p^a q^b s^{a+b}}{p^a q^b s^{a+b} + (1-(p+q)s)\sum_{j=0}^{k} p^{c_j} q^{d_j} s^{c_j+d_j}}.$$

The "nonhomogenised" form, obtained by replacing the $(p+q)$ factor by 1, is given in slightly disguised form as [1, exercise 20, chapter 8].

5. A recurrence relation for the $c(n,i)$

Even though we defined the coefficients $c(n,i)$ above for $n \geq 0$ and i satisfying $0 \leq i \leq n$, it is convenient to define $c(n,i)$ for all integers n and i.

Define $c(n,i) = 0$ if $n < 0$ or if $n \geq 0$ and either $i < 0$ or $i > n$. With this extension of the definition of $c(n,i)$ we can prove the following recurrence relation.

Proposition 3. *The coefficients $c(n,i)$ satisfy $c(0,0) = 1$ and for $n > 0$*

$$\begin{aligned} c(n,i) = c(n-1,i) &+ c(n-1,i-1) \\ &+ \sum_{j=1}^{k} c(n-1-c_j-d_j, i-d_j) \\ &+ \sum_{j=1}^{k} c(n-1-c_j-d_j, i-d_j-1) \\ &- \sum_{j=1}^{k} c(n-c_j-d_j, i-d_j) - c(n-a-b, i-b). \end{aligned}$$

for every $i \in \mathbb{Z}$.

PROOF. Clearly $c(0,0) = 1$. So assume $n > 0$. With our above extension of the definition of the coefficients $c(n,i)$, we can substitute $\sum_{i=-\infty}^{\infty} c(m,i)p^{m-i}q^i$ for each occurrence of $f_\pi(m)$ in 4.1 and, after canceling p^aq^b, we get

$$\begin{aligned} \sum_{i=-\infty}^{\infty} c(n,i)p^{n-i}q^i = \sum_{i=-\infty}^{\infty} & c(n-1,i)p^{n-1-i}q^i(p+q) \\ &+ \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-1-c_j-d_j,i)p^{n-1-c_j-d_j-i}q^i(p+q)p^{c_j}q^{d_j} \\ &- \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-c_j-d_j,i)p^{n-c_j-d_j-i}q^ip^{c_j}q^{d_j} \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{i=-\infty}^{\infty} c(n-a-b,i)p^{n-a-b-i}q^i p^a q^b \\
&= \sum_{i=-\infty}^{\infty} c(n-1,i)p^{n-i}q^i + \sum_{i=-\infty}^{\infty} c(n-1,i)p^{n-1-i}q^{i+1} \\
&\quad + \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-1-c_j-d_j,i)p^{n-d_j-i}q^{i+d_j} \\
&\quad + \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-1-c_j-d_j,i)p^{n-1-d_j-i}q^{i+d_j+1} \\
&\quad - \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-c_j-d_j,i)p^{n-d_j-i}q^{i+d_j} \\
&\quad - \sum_{i=-\infty}^{\infty} c(n-a-b,i)p^{n-b-i}q^{i+b} \\
&= \sum_{i=-\infty}^{\infty} c(n-1,i)p^{n-i}q^i + \sum_{i=-\infty}^{\infty} c(n-1,i-1)p^{n-i}q^{i} \\
&\quad + \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-1-c_j-d_j,i-d_j)p^{n-i}q^{i} \\
&\quad + \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-1-c_j-d_j,i-d_j-1)p^{n-i}q^{i} \\
&\quad - \sum_{j=1}^{k}\sum_{i=-\infty}^{\infty} c(n-c_j-d_j,i-d_j)p^{n-i}q^{i} \\
&\quad - \sum_{i=-\infty}^{\infty} c(n-a-b,i-b)p^{n-i}q^{i}
\end{aligned}
$$

Reading off the coefficient of $p^{n-i}q^i$ on each side proves the result. ■

6. The generating function for the $c(n,i)$

Proposition 4. *The generating function $g_\pi(t,z)$ for the $c(n,i)$ is*

$$
g_\pi(t,z) = \frac{1}{(1-t-tz)\sum_{j=0}^{k} t^{c_j+d_j}z^{d_j} + t^{a+b}z^b}.
$$

PROOF. Since $c(n,i)=0$ when $n<0$, we can write $g_\pi(t,z) = \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n,i)t^n z^i$.

Now

$$
\begin{aligned}
t^r z^s g_\pi(t,z) &= \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n,i)t^{n+r}z^{i+s} \\
&= \sum_{n\geq a}\sum_{i=-\infty}^{\infty} c(n-r,i-s)t^n z^i \\
&= \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-r,i-s)t^n z^i
\end{aligned}
$$

since $c(n-r,\ell)=0$ if $n-r<0$.

Therefore

$$
\begin{aligned}
(1 \;-\; & t - tz + \sum_{j=1}^{k} t^{c_j+d_j}z^{d_j} - \sum_{j=1}^{k} t^{c_j+d_j+1}z^{d_j} - \sum_{j=1}^{k} t^{c_j+d_j+1}z^{d_j+1} + t^{a+b}z^b)g_\pi(t,z) \\
= \; & \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n,i)t^n z^i - \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-1,i)t^n z^i - \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-1,i-1)t^n z^i \\
& \quad + \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-c_i-d_i,i-d_i)t^n z^i \\
& \quad - \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-c_i-d_i-1,i-d_i)t^n z^i \\
& \quad - \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-c_i-d_i-1,i-d_i-1)t^n z^i \\
& \quad + \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} c(n-a-b,i-b)t^n z^i \\
= \; & \sum_{n\geq 0}\sum_{i=-\infty}^{\infty} (c(n,i) - c(n-1,i) - c(n-1,i-1) + c(n-c_i-d_i,i-d_i) \\
& \qquad\qquad - c(n-c_i-d_i-1,i-d_i) - c(n-c_i-d_i-1,i-d_i-1) \\
& \qquad\qquad + c(n-a-b,i-b))t^n z^i \\
= \; & c(0,0) + \sum_{n\geq 1}\sum_{i=-\infty}^{\infty} (0)t^n z^i, \quad \text{[by Proposition 3]} \\
= \; & 1.
\end{aligned}
$$

■

7. Two special cases

Before we a derive a formula for $c(n,i)$ in the general case, we do so in two special cases also considered in [4].

First we look at the case $\pi = \underbrace{SS\ldots S}_{a}$, consisting of a Ss in a row. Here $k = a-1$, $b_i = 0$ for $i = 0, \ldots, k$. So $c_i = i$ for $i = 0, \ldots, k$. Hence

$$g_\pi(t,z) = \frac{1}{(1-t-tz)\sum_{i=0}^{a-1} t^i + t^a} \tag{7.1}$$

$$= \frac{1}{1-tz-t^2z-\ldots-t^a z} \tag{7.2}$$

$$= \frac{1-t}{1-t-tz+t^{a+1}z}. \tag{7.3}$$

Working with expression 7.3 for $g_\pi(t,z)$, we see that

$$\begin{aligned} g_\pi(t,z) &= (1-t)\sum_{j=0}^{\infty} t^j (1+z-t^a z)^j \\ &= (1-t)\sum_{j\geq 0} t^j \sum_{n_1+n_2+n_3=j} \binom{j}{n_1,n_2,n_3} z^{n_2}(-1)^{n_3} t^{n_3 a} z^{n_3} \\ &= \sum_{j\geq 0}\sum_{n_1+n_2+n_3=j} \binom{j}{n_1,n_2,n_3} (-1)^{n_3} t^{n_3 a+j} z^{n_2+n_3} \\ &\qquad - \sum_{j\geq 0}\sum_{n_1+n_2+n_3=j} \binom{j}{n_1,n_2,n_3} (-1)^{n_3} t^{n_3 a+j+1} z^{n_2+n_3} \end{aligned}$$

Now $c(n,i)$ is the coefficient of $t^n z^i$ in $g_\pi(t,z)$. To compute the coefficient of $t^n z^i$ in the first of the two terms on the right of the previous expression, we set $n = n_3 a + j$ and $i = n_2 + n_3$. So j takes the values $n - n_3 a$ as n_3 takes the values 0 to $\lfloor n/a_0 \rfloor$. So $n_2 = i - n_3$ and $n_1 = j - n_2 - n_3 = n - n_3 a - (i - n_3) - n_3 = n - n_3 a - i$. Replacing n_3 by k we see that the coefficient of $t^n z^i$ in the first term is

$$\sum_{k=0}^{\lfloor n/a \rfloor} (-1)^k \binom{n-ka}{n-i-ka, i-k, k}.$$

To compute the coefficient of $t^n z^i$ in the second term, we solve the equations $n = n_3 a + j + 1$ and $i = n_2 + n_3$ for n_1, n_2, and n_3, yielding this coefficient as

$$-\sum_{k=0}^{\lfloor (n-1)/a \rfloor} (-1)^k \binom{n-1-ka}{n-1-i-ka, i-k, k}.$$

Thus

$$c(n,i) = \sum_{k=0}^{\lfloor n/a \rfloor} (-1)^k \binom{n-ka}{n-i-ka, i-k, k} - \sum_{k=0}^{\lfloor (n-1)/a \rfloor} (-1)^k \binom{n-1-ka}{n-1-i-ka, i-k, k}.$$

Starting from expression 7.2

$$g_\pi(t,z) = \frac{1}{1-tz-t^2z-\ldots-t^az},$$

we can recover the 1982 result of Philippou and Muwafi [5] that

$$c(n,i) = \sum \binom{i}{n_1, n_2, \ldots, n_a}, \quad \text{for the pattern } \pi = \underbrace{SS\ldots S}_{a},$$

where the summation is over all a-tuples $(n_1, \ldots, n_a)$ of nonnegative integers satisfying

$$n_1 + n_2 + \ldots + n_a = i$$
$$n_2 + 2n_3 + \ldots + (a-1)n_a = n-i.$$

This result is more like the expression of $c(n,i)$ for general π that we will derive in the next section.

When $\pi = \underbrace{FF\ldots F}_{b}$, consisting of b Fs in a row,

$$g_\pi(t,z) = \frac{1}{1-t-t^2z-t^3z^2-\ldots-t^bz^{b-1}} = \frac{1-tz}{1-t-tz+t^{b+1}z^b}.$$

Using techniques similar to those above, one can show that

$$c(n,i) = \sum_{k=0}^{\lfloor n/b \rfloor} (-1)^k \binom{n-bk}{n-i-k, i-bk, k} - \sum_{k=0}^{\lfloor (n-1)/b \rfloor} (-1)^k \binom{n-bk-1}{n-i-k, i-bk-1, k}$$

for the pattern $\pi = \underbrace{FF\ldots F}_{b}$.

Next we look at the case of aperiodic π. Then $k=0$ and

$$g_\pi(t,z) = \frac{1}{1-t-tz+t^{a+b}z^b}.$$

It is easy to show that if $a+b>1$, then

$$c(n,i) = \sum_{k=0}^{\lfloor n/(a+b-1) \rfloor} (-1)^k \binom{n-k(a+b-1)}{n-i-ka, i-bk, k} \quad \text{for aperiodic } \pi.$$

8. Two formulas for $c(n,i)$

Theorem 1. *For* $0 \le i \le n$,

$$c(n,i) = \sum \binom{\sum_{j=1}^{3} n_j + \sum_{j=1}^{k} m_j + \sum_{j=1}^{k} p_j + \sum_{j=1}^{k} q_j}{n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k} (-1)^{n_3 + \sum_{j=1}^{k} m_j}$$

where the outer summation is over all $(3k+3)$*-tuples*

$$(n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k)$$

of nonnegative integers satisfying

$$n_1 + an_3 + \sum_{j=1}^{k} c_j m_j + \sum_{j=1}^{k} (c_j+1) p_j + \sum_{j=1}^{k} c_j q_j = n - i$$
$$n_2 + bn_3 + \sum_{j=1}^{k} d_j m_j + \sum_{j=1}^{k} d_j p_j + \sum_{j=1}^{k} (d_j+1) q_j = i.$$

Proof.

$$\begin{aligned}
g_\pi(t,z) &= \frac{1}{(1-t-tz)\sum_{j=0}^{k} t^{c_j+d_j} z^{d_j} + t^{a+b} z^b} \\
&= \frac{1}{1-t-tz+(1-t-tz)\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j} + t^{a+b} z^b} \\
&= \frac{1}{1-t-tz+t^{a+b}z^b + \sum_{j=1}^{k} t^{c_j+d_j} z^{d_j} - \sum_{j=1}^{k} t^{c_j+d_j+1} z^{d_j} - \sum_{j=1}^{k} t^{c_j+d_j+1} z^{d_j+1}} \\
&= \sum_{\ell=0}^{\infty} \left(t + tz - t^{a+b} z^b - \sum_{j=1}^{k} t^{c_j+d_j} z^{d_j} + \sum_{j=1}^{k} t^{c_j+d_j+1} z^{d_j} + \sum_{j=1}^{k} t^{c_j+d_j+1} z^{d_j+1} \right)^{\ell} \\
&= \sum_{\ell=0}^{\infty} \sum_{\sum n_j + \sum m_j + \sum p_j + \sum q_j = \ell} \binom{\ell}{n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k} \\
&\quad \times t^{n_1} (tz)^{n_2} (-t^{a+b} z^b)^{n_3} \prod (-t^{c_j+d_j} z^{d_j})^{m_j} \prod (t^{c_j+d_j+1} z^{d_j})^{p_j} \prod (t^{c_j+d_j+1} z^{d_j+1})^{q_j} \\
&= \sum_{\ell=0}^{\infty} \sum_{\sum n_j + \sum m_j + \sum p_j + \sum q_j = \ell} \binom{\ell}{n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k} \\
&\quad \times (-1)^{n_3 + \sum m_j} t^{n_1 + n_2 + (a+b) n_3 + \sum (c_j+d_j) m_j + \sum (c_j+d_j+1) p_j + \sum (c_j+d_j+1) q_j} \\
&\quad \times z^{n_2 + bn_3 + \sum d_j m_j + \sum d_j p_j + \sum (d_j+1) q_j}
\end{aligned}$$

Hence, if $0 \leq i \leq n$, the coefficient of $t^n z^i$ is

$$\sum \binom{\sum n_j + \sum m_j + \sum p_j + \sum q_j}{n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k} (-1)^{n_3 + \sum m_j}$$

where the left-most summation is over all $(3k+3)$-tuples

$$(n_1, n_2, n_3, m_1, \ldots, m_k, p_1, \ldots, p_k, q_1, \ldots, q_k)$$

of nonnegative integers satisfying

$$n_1 + n_2 + (a+b)n_3 + \sum (c_j + d_j) m_j + \sum (c_j + d_j + 1) p_j + \sum (c_j + d_j + 1) q_j = n$$
$$n_2 + bn_3 + \sum d_j m_j + \sum d_j p_j + \sum (d_j + 1) q_j = i,$$

or

$$n_1 + an_3 + \sum c_i m_i + \sum (c_j + 1) p_j + \sum c_j q_j = n - i$$
$$n_2 + bn_3 + \sum d_j m_j + \sum d_j p_j + \sum (d_j + 1) q_j = i.$$

■

Can we get by with fewer than $3k+3$ variables? We saw that when π consists of a H's, then a variables are used in the expression for $c(n,i)$ of Philippou and Muwafi. One way we can get by with fewer variables is by using integer inequalities rather than equalities.

Theorem 2. *For* $0 \leq i \leq n$,

$$c(n,i) = \sum \binom{n_1 + n_2 + n_3 + \sum_{j=1}^k e_j}{n_1, n_2, n_3, e_1, \ldots, e_k} (-1)^{n_3}$$

where the outer summation is over all $(k+3)$*-tuples* $(n_1, n_2, n_3, e_1, \ldots, e_k)$ *of nonnegative integers satisfying*

$$n_1 + an_3 + \sum_{j=1}^k c_j e_j \leq n - i$$
$$n_2 + bn_3 + \sum_{j=1}^k d_j e_j \leq i$$
$$n \leq n_1 + n_2 + (a+b)n_3 + \sum_{j=1}^k (c_j + d_j + 1) e_j.$$

PROOF.

$$
\begin{aligned}
g_\pi(t,z) &= \frac{1}{(1-t-tz)\sum_{j=0}^{k} t^{c_j+d_j} z^{d_j} + t^{a+b}z^b} \\
&= \frac{1}{1-t-tz+(1-t-tz)\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j} + t^{a+b}z^b} \\
&= \frac{1}{1-t-tz+t^{a+b}z^b+(1-t-tz)\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j}} \\
&= \sum_{\ell=0}^{\infty}\left(t+tz-t^{a+b}z^b+(t+tz-1)\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j}\right)^{\ell} \\
&= \sum_{\ell=0}^{\infty}\sum_{\sum n_u=\ell}\binom{\ell}{n_1,n_2,n_3,n_4} t^{n_1}(tz)^{n_2}(-t^{a+b}z^b)^{n_3}\left((t+tz-1)\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j}\right)^{n_4}.
\end{aligned}
$$

Now

$$
(t+tz-1)^{n_4} = \sum_{\sum m_v=n_4}\binom{n_4}{m_1,m_2,m_3} t^{m_1} t^{m_2} z^{m_2}(-1)^{m_3}
$$

and

$$
\left(\sum_{j=1}^{k} t^{c_j+d_j} z^{d_j}\right)^{n_4} = \sum_{\sum e_j=n_4}\binom{n_4}{e_1,\dots,e_k}\prod_{j=1}^{k} t^{(c_j+d_j)e_j} z^{d_j\ell_j}.
$$

So $g_\pi(t,z)$ is equal to

$$
\sum_{\ell=0}^{\infty}\sum_{\sum n_u=\ell}\binom{\ell}{n_1,n_2,n_3,n_4}\sum_{\sum m_v=n_4}\binom{n_4}{m_1,m_2,m_3}\sum_{\sum e_j=n_4}\binom{n_4}{e_1,\dots,e_k}
$$
$$
\times(-1)^{n_3+m_3} t^{n_1+n_2+(a+b)n_3+m_1+m_2+\sum_{j=1}^{k}(c_j+d_j)e_j} z^{n_2+bn_3+m_2+\sum_{j=1}^{k} d_j e_j}
$$

where

$$
\begin{aligned}
\ell &= n_1+n_2+n_3+n_4 \\
\text{and } n_4 &= m_1+m_2+m_3 = \sum_{j=1}^{k} e_i.
\end{aligned}
$$

Hence, if $0 \le i \le n$, the coefficient of $t^n z^i$ is

$$
\sum\binom{n_1+n_2+n_3+\sum e_j}{n_1,n_2,n_3,e_1,\dots,e_k}(-1)^{n_3}\sum_{m_1+m_2+m_3=\sum e_j}\binom{\sum e_j}{m_1,m_2,m_3}(-1)^{m_3}
$$

where the leftmost summation is over all $(k+6)$-tuples $(n_1, n_2, n_3, e_1, \ldots, e_k, m_1, m_2, m_3)$ of nonnegative integers satisfying

$$n_1 + an_3 + m_1 + \sum c_j e_j = n - i$$
$$n_2 + bn_3 + m_2 + \sum d_j e_j = i$$
$$m_1 + m_2 + m_3 = \sum e_j.$$

Note that

$$\sum_{m_1+m_2+m_3=\sum e_j} \binom{\sum e_j}{m_1, m_2, m_3} (-1)^{m_3} = (1+1-1)^{\sum e_j} = 1.$$

The variables m_1, m_2, and m_3 can be eliminated. Then the coefficient of $t^n z^i$ is

$$\sum \binom{n_1 + n_2 + n_3 + \sum e_j}{n_1, n_2, n_3, e_1, \ldots, e_k} (-1)^{n_3}$$

where the summation is over all $(k+3)$-tuples $(n_1, n_2, n_3, e_1, \ldots, e_k)$ of nonnegative integers satisfying

$$n_1 + an_3 + \sum c_j e_j \leq n - i$$
$$n_2 + bn_3 + \sum d_j e_j \leq i$$
$$n \leq n_1 + n_2 + (a+b)n_3 + \sum (c_j + d_j + 1) e_j.$$

■

9. Unimodality of the $c(n, i)$

We conjecture that for fixed $n \geq 0$, the sequence

$$c(n, 0), c(n, 1), \ldots, c(n, n)$$

is unimodal.

In random walk theory, there is an interpretation of $c(n, i)$ as the number of northeast walks from $(-a_0, -b_0)$ to $(n-i, i)$ that contain a certain pattern of Ns and Es associated in an obvious way with π in the part of the walk from $(-a_0, -b_0)$ to $(0, 0)$ and which subsequently avoid that pattern.

In general, this sequence is not symmetric, not log-concave, some of the $c(n, i)$ can equal 0, and the polynomial $\sum_{i=0}^{n} c(n, i) x^i$ can have complex roots. However when π is aperiodic, the sequence consists of positive terms and we conjecture that the sequence is log-concave.

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