DETERMINANTS OF CHARACTERISTIC FUNCTIONS OF P-HYPONORMAL OPERATORS

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Dedicated to Professor Vladimir Müller on his sixtieth birthday

Abstract

We introduce characteristic functions for p-hyponormal operators and calculate their determinants.

1. Introduction

Professor D. Xia, in [12], calculated the determinants of characteristic functions of semi-hyponormal operators. In this paper we calculate the determinants of characteristic functions of p-hyponormal operators for $0 < p < \frac{1}{2}$.

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, T is called hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. The set of all semi-hyponormal operators in $B(\mathcal{H})$ is denoted by SH. The set of all p-hyponormal operators in $B(\mathcal{H})$ is denoted by p-H. Let SHU and p-HU denote the sets of all operators in SH and in p-H with equal defect and nullity (cf. [11, p. 4]), respectively. Hence we may assume that the operator U in the polar decomposition $T = U|T|$ is unitary if $T \in$SHU $\cup$ p-HU. Throughout this paper, we assume that p is in the interval $0 < p < \frac{1}{2}$.

Let A be a contraction and $T \in B(\mathcal{H})$. Denote

$$A^{[n]} = \begin{cases} A^n, & n \geq 0 \\ (A^*)^{-n}, & n < 0. \end{cases}$$

If

$$\mathcal{S}_A^{\pm}(T) = \text{s -} \lim_{n \to \pm\infty} A^{[-n]} T A^{[n]}$$

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exist, then operators $\mathcal{S}_A^{\pm}(T)$ are called polar symbols of T related to A. If an operator $T = U|T|$ is semi-hyponormal, then $\mathcal{S}_U^{\pm}(T)$ exist and, for $0 \leq k \leq 1$,

$$T_{(k)} = (1-k)\mathcal{S}_U^{-}(T) + k\mathcal{S}_U^{+}(T)$$

are called the general polar symbols of T. Then the following property holds: If $T = U|T|$ with unitary U is p-hyponormal, then $\mathcal{S}_U^{-}(|T|^{2p}) \leq |T|^{2p} \leq \mathcal{S}_U^{+}(|T|^{2p})$, because $U|T|^{2p}$ is a semi-hyponormal operator. See Xia ([12, p. 37]) for details.

Let $\mathbf{T} = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$, Σ be the set of all Borel sets in $\mathbf{T}$ and let m be a measure on the measurable space $(\mathbf{T}, \Sigma)$ such that $dm(e^{i\theta}) = \frac{1}{2\pi} d\theta$ and $\mathcal{D}$ be a separable Hilbert space.

The Hilbert space of all vector-valued, strongly measurable and square-integrable functions with values in $\mathcal{D}$ and with inner product

$$(f,g) = \int_{\mathbf{T}} (f(e^{i\theta}), g(e^{i\theta}))_{\mathcal{D}} dm$$

is denoted by $L^2(\mathcal{D})$; Hardy space is denoted by $H^2(\mathcal{D})$, and the projection from $L^2(\mathcal{D})$ to $H^2(\mathcal{D})$, by $\mathcal{P}$. If $f \in L^2(\mathcal{D})$, then

$$(\mathcal{P}(f))(e^{i\theta}) = \lim_{r \to 1-0} \frac{1}{2\pi i} \int_{|z|=1} f(z)(z - re^{i\theta})^{-1} dz.$$

Let ν be a singular measure on $(\mathbf{T}, \Sigma)$, $F \in \Sigma$ be a set such that $\nu(\mathbf{T} \setminus F) = 0$ and $m(F) = 0$. Put $\mu = m + \nu$. Let $R(\cdot)$ be a standard operator-valued strongly measurable function defined on $\Omega = (\mathbf{T}, \Sigma, \mu)$ with values being the projection in $\mathcal{D}$, $L^2(\Omega, \mathcal{D})$ be a Hilbert space of all $\mathcal{D}$-valued, strongly measurable and square-integrable functions on Ω with inner product $(f,g) = \int_{\mathbf{T}} (f(e^{i\theta}), g(e^{i\theta}))_{\mathcal{D}} d\mu$, and

$$\widehat{H} = \{f \mid f \in L^2(\Omega, \mathcal{D}), R(e^{i\theta}) f(e^{i\theta}) = f(e^{i\theta}),\ e^{i\theta} \in \mathbf{T}\}.$$

Then $\widehat{H}$ is a subspace of $L^2(\Omega, \mathcal{D})$. The space $L^2(\mathcal{D})$ is identified with a subspace of $L^2(\Omega, \mathcal{D})$. Hence $\mathcal{P}$ extends to $L^2(\Omega, \mathcal{D})$ such that

$$\mathcal{P}f = 0 \quad \text{for} \quad f \in L^2(\Omega, \mathcal{D}) \ominus L^2(\mathcal{D}).$$

We define an operator $\mathcal{P}_0$ from $L^2(\Omega, \mathcal{D})$ to $\mathcal{D}$ as follows:

$$\mathcal{P}_0(f) = \int f(e^{i\theta}) dm.$$

Then $\mathcal{P}_0$ is the projection from $L^2(\Omega, \mathcal{D})$ to $\mathcal{D}$ (cf. [12, p. 50]). Let $\alpha(\cdot)$ and $\beta(\cdot)$ be operator-valued, uniformly bounded and strongly measurable functions on Ω such that $\alpha(e^{i\theta})$ and $\beta(e^{i\theta})$ are linear operators in $\mathcal{D}$, satisfying

$$R(e^{i\theta})\alpha(e^{i\theta}) = \alpha(e^{i\theta})R(e^{i\theta}) = \alpha(e^{i\theta}),\ R(e^{i\theta})\beta(e^{i\theta}) = \beta(e^{i\theta})R(e^{i\theta}) = \beta(e^{i\theta})$$

and $\beta(e^{i\theta}) \geq 0$.

Furthermore, suppose that $\alpha(e^{i\theta}) = 0$ if $e^{i\theta} \in F$. And we denote $(\alpha f)(e^{i\theta}) = \alpha(e^{i\theta})f(e^{i\theta})$.

An operator $\widehat{U}$ in $\widehat{\mathcal{H}}$ is defined by

$$(\widehat{U}f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta}).$$

Let $T = U|T| \in p$-HU. The general polar symbols $T_{[k]}$ of T are defined by

$$T_{[k]} = U\{(1-k)\mathcal{S}_U^-(|T|^{2p}) + k\mathcal{S}_U^+(|T|^{2p})\}^{\frac{1}{2p}}$$

(see [4] and [5] for details).

Since $\beta(e^{i\theta}) \geq 0$ and $\mathcal{P}$ is a projection on $L^2(\mathcal{D})$, we have $(\alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta}), f(e^{i\theta}))_{\mathcal{D}} \geq 0$. Therefore, we can define the operator $\{\alpha^*\mathcal{P}\alpha + \beta\}^{\frac{1}{2p}}$. And we showed the following results.

Theorem A ([4, theorem 1]). *With the above notations, let $\widetilde{T}$ be an operator in $\widehat{\mathcal{H}}$ defined by*

$$(\widetilde{T}f)(e^{i\theta}) = e^{i\theta}(Af)(e^{i\theta}),$$

where $(A^{2p}f)(e^{i\theta}) = \alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta})$. *Then* $\widetilde{T}$ is p-hyponormal and the corresponding polar symbols of $\widetilde{T}$ are

$$(\mathcal{S}_{\widehat{U}}^+(\widetilde{T})f)(e^{i\theta}) = e^{i\theta}\{\alpha(e^{i\theta})^*\alpha(e^{i\theta}) + \beta(e^{i\theta})\}^{\frac{1}{2p}}f(e^{i\theta})$$

and

$$(\mathcal{S}_{\widehat{U}}^-(\widetilde{T})f)(e^{i\theta}) = e^{i\theta}\{\beta(e^{i\theta})\}^{\frac{1}{2p}}f(e^{i\theta}).$$

The corresponding polar difference operator $\widetilde{Q}_p = |\widetilde{T}|^{2p} - \widehat{U}|\widetilde{T}|^{2p}\widehat{U}^*$ is

$$(\widetilde{Q}_pf)(e^{i\theta}) = \alpha(e^{i\theta})^*\big(\mathcal{P}_0(\alpha f)\big)(e^{i\theta}).$$

Moreover, the corresponding general polar symbols $\widetilde{T}_{[k]}$ are

$$(\widetilde{T}_{[k]}f)(e^{i\theta}) = e^{i\theta}\{\beta(e^{i\theta}) + k\alpha(e^{i\theta})^*\alpha(e^{i\theta})\}^{\frac{1}{2p}}$$

Theorem B ([4, theorem 3]). *Let $T = U|T|$ be a p-hyponormal operator in $\mathcal{H}$ such that U is unitary. Then there exists a function space $\widehat{\mathcal{H}}$, and operators $\widetilde{T}$ and $\widehat{U}$ in $\widehat{\mathcal{H}}$ which have the forms in Theorem A such that*

$$WTW^{-1} = \widetilde{T} \text{ and } WUW^{-1} = \widehat{U},$$

where W is a unitary operator from $\mathcal{H}$ to $\widehat{\mathcal{H}}$. Moreover $\alpha(\cdot) \geq 0$.

2. Determinants of characteristic functions

By Theorems A and B, we may only consider the singular integral model $\widetilde{T} = \widehat{U}|\widetilde{T}|$ of a p-hyponormal operator $T = U|T|$. Hence for the instance we use T and U instead of $\widetilde{T}$ and $\widehat{U}$, respectively. In particular, from Theorem B, we may assume that α is positive. For an operator S, $\rho(S)$, $\sigma(S)$, $\sigma_a(S)$ and $\sigma_p(S)$ denote the resolvent set, the spectrum, the approximate point spectrum and the point spectrum of S, respectively.

Here we write $|T|_{\pm}^{2p} = \mathcal{S}_U^{\pm}(|T|^{2p})$ for the instance. Then $\mathcal{S}_U^{\pm}(U|T|^{2p}) = U|T|_{\pm}^{2p}$. Then it is known that $U|T|_{\pm}$ are normal operators and the following result holds.

Theorem C ([12, theorem II.1.5]). *With the above notations, it holds that*

$$\sigma(U|T|_{\pm}^{2p}) \subset \sigma(U|T|^{2p}) \text{ and } \sigma_a(U|T|_{\pm}^{2p}) \subset \sigma_a(U|T|^{2p}).$$

If $T = U|T| \in p$-HU, then it is easy to see that

$$\sigma(U|T|_{\pm}) \subset \sigma(U|T|) \text{ and } \sigma_a(U|T|_{\pm}) \subset \sigma_a(U|T|).$$

Also by [4] we have

$$(|T|_{+}^{2p} f)(e^{i\theta}) = \left(\alpha(e^{i\theta})^2 + \beta(e^{i\theta})\right) f(e^{i\theta}) \quad \text{and} \quad (|T|_{-}^{2p} f)(e^{i\theta}) = \beta(e^{i\theta}) f(e^{i\theta}).$$

Therefore, we have

$$\alpha^2 = |T|_{+}^{2p} - |T|_{-}^{2p}. \tag{2.1}$$

Definition 1. For $\ell \in \rho(U|T|_{-}^{2p})$, we define $W_\ell(e^{i\theta})$ as follows:

$$W_\ell(e^{i\theta}) = I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell e^{-i\theta})^{-1}\alpha(e^{i\theta}).$$

The operator $W_\ell(e^{i\theta})$ is called the characteristic function of T. We remark that if $\ell \in \rho(U|T|_{-}^{2p})$, then $\ell e^{-i\theta} \in \rho(\beta(e^{i\theta}))$.

We define an operator $\widetilde{W_\ell}$ on $H^2(\mathcal{D})$ as follows:

$$\left(\widetilde{W_\ell} f\right)(\cdot) = \mathcal{P}\left(W_\ell(\cdot) f(\cdot)\right).$$

We call $\widetilde{W_\ell}$ the Toeplitz operator corresponding to W_ℓ. Since $U|T|^{2p}$ $\in$SHU, by

Theorems V.5.1 and 5.2 of [12] we have the following two results. Hence we omit the proof.

Theorem 1. *Let* $T = U|T| \in p\text{-HU}$ *and* $\ell \in \rho(U|T|_-^{2p})$. *Then*

$$\ell \in \sigma_p(U|T|^{2p}) \quad (\textit{or} \quad \bar{\ell} \in \sigma_p((U|T|^{2p})^*))$$

if and only if

$$0 \in \sigma_p(\widetilde{W_\ell}) \quad (\textit{correspondingly } 0 \in \sigma_p((\widetilde{W_\ell})^*).$$

The restriction of the mapping

$$L_\ell : h \longrightarrow (\ell e^{-i\theta} - \beta)^{-1} \alpha h, \quad (h \in H^2(\mathcal{D}))$$

to $\ker(\widetilde{W_\ell})$ *is a one-to-one bounded linear operator from* $\ker(\widetilde{W_\ell})$ *to* $\ker(U|T|^{2p} - \ell)$, *whose inverse is* $f \longrightarrow \mathcal{P}(\alpha f)$ (*correspondingly*

$$M_\ell : h \longrightarrow (\bar{\ell} e^{i\theta} - \beta)^{-1} \alpha h, \quad (h \in \ker((\widetilde{W_\ell})^*)$$

is a bounded linear operator from $\ker((\widetilde{W_\ell})^*)$ *to* $\ker((U|T|^{2p})^* - \ell)$).

Let $\ell \in \sigma_p(U|T|^{2p})$ (*respectively* $\bar{\ell} \in \sigma_p((U|T|^{2p})^*)$). *Then the multiplicity of* ℓ (*respectively* $\bar{\ell}$) *is equal to the multiplicity of the eigenvalue* 0 *of the operator* $\widetilde{W_\ell}$ (*respectively* $(\widetilde{W_\ell})^*$).

Using the Berberian method, we construct $\mathcal{R} \supset \mathcal{H}$ and $\mathcal{R}^2(\mathcal{D}) \supset H^2(\mathcal{D})$. The mappings $B(\mathcal{H}) \longrightarrow B(\mathcal{R})$ and $B(H^2(\mathcal{D})) \longrightarrow B(\mathcal{R}^2(\mathcal{D}))$ are both denoted by π. Then we have

$$\sigma_a(T) = \sigma_p(\pi(T)).$$

The dimension of $\ker(\pi(T - \ell))$ is called the approximate multiplicity of T corresponding to the approximate eigenvalue ℓ.

Theorem 2. *Let* $T = U|T| \in p\text{-HU}$ *and* $\ell \in \rho(U|T|_-^{2p})$. *Then*

$$\ell \in \sigma_a(U|T|^{2p}) \quad (\textit{or} \quad \bar{\ell} \in \sigma_a((U|T|^{2p})^*))$$

if and only if

$$0 \in \sigma_a(\widetilde{W_\ell}) \quad (\textit{correspondingly } 0 \in \sigma_a((\widetilde{W_\ell})^*).$$

The mapping $\pi(L_\ell)$ (*or* $\pi(M_\ell)$) *is a one-to-one bounded linear operator from* $\ker(\pi(\widetilde{W_\ell}))$ *to* $\ker(\pi(U|T|^{2p} - \ell))$ (*respectively from* $\ker(\pi((\widetilde{W_\ell})^*))$ *to* $\ker(\pi((U|T|^{2p})^* - \bar{\ell}))$ *and its inverse operator is bounded.*

Let $\ell \in \sigma_a(U|T|^{2p})$ (*respectively* $\bar{\ell} \in \sigma_p((U|T|^{2p})^*)$). *Then the approximate multiplicity of* ℓ (*respectively* $\bar{\ell}$) *is equal to the approximate multiplicity of the eigenvalue* 0 *of the operator* $\widetilde{W_\ell}$ (*respectively* $(\widetilde{W_\ell})^*$).

For the next theorem, we need the following result.

Theorem D ([2, theorem 4]). *Let $T = U|T|$ be p-hyponormal. If $(T - re^{i\theta})f_n \longrightarrow 0$, then $(|T| - r)f_n \longrightarrow 0$ and $(U - e^{i\theta})f_n \longrightarrow 0$.*

Theorem 3. (1) *Let $T = U|T| \in p$-HU and $\ell = |\ell|e^{i\omega} \in \rho(U|T|_-^{2p})$. If $\{h_n\}$ is a sequence of unit vectors in $H^2(\mathcal{D})$ such that $\widetilde{W_\ell}h_n \longrightarrow 0$, then $|\ell|^{\frac{1}{2p}}e^{i\omega} \in \sigma_a(T)$ and*
$\|(T - |\ell|^{\frac{1}{2p}}e^{i\omega})f_n\| \longrightarrow 0$, where $f_n = \dfrac{L_\ell h_n}{\|L_\ell h_n\|}$ for sufficiently large n.

(2) *If $\ell \in \sigma_a(T)$, $\ell_p \in \rho(U|T|_-^{2p})$ and $\{f_n\}$ is a sequence of unit vectors in $\mathcal{H}$ such that $\|(T - \ell)f_n\| \longrightarrow 0$, then there must be a sequence $\{h_n\}$ of unit vectors in $H^2(\mathcal{D})$ such that $\widetilde{W_{\ell_p}}h_n \longrightarrow 0$ and $\|f_n - \dfrac{L_{\ell_p}h_n}{\|L_{\ell_p}h_n\|}\| \longrightarrow 0$, where $\ell_p = |\ell|^{2p}e^{i\omega}$.*

Proof.
(1) Since $(U|T|^{2p} - l)L_\ell h_n = -e^{i\theta}\widetilde{W_\ell}lh_n$, we have $(U|T|^{2p} - l)L_\ell h_n \longrightarrow 0$. Since $\alpha L_\ell h_n = (I - \widetilde{W_\ell})h_n$, it holds that $\|(U|T|^{2p} - \ell)f_n\| \longrightarrow 0$, where $f_n = \dfrac{L_\ell h_n}{\|L_\ell h_n\|}$ for sufficiently large n. Since $U|T|^{2p}$ is semi-hyponormal, by Theorem D we have $\|(U - e^{i\omega})f_n\| \longrightarrow 0$ and $\|(|T|^{2p} - |\ell|)f_n\| \longrightarrow 0$. Hence it is clear that $\|(|T| - |\ell|^{\frac{1}{2p}})f_n\| \longrightarrow 0$. Hence we have $|\ell|^{\frac{1}{2p}}e^{i\omega} \in \sigma_a(T)$.

(2) Let $\|(T - \ell)f_n\| \longrightarrow 0$. Then by Theorem D we have $\|(|T| - |\ell|)f_n\| \longrightarrow 0$ and $\|(U - e^{i\omega})f_n\| \longrightarrow 0$. It is easy to see that $\|(U|T|^{2p} - |\ell|^{2p}e^{i\omega})f_n\| \longrightarrow 0$. Since $\widetilde{W_{\ell_p}}\mathcal{P}\alpha = \mathcal{P}\alpha(U|T|_-^{2p} - |\ell|^{2p}e^{i\omega})^{-1}(U|T|^{2p} - |\ell|^{2p}e^{i\omega})$, we have $\widetilde{W_{\ell_p}}\mathcal{P}\alpha f_n \longrightarrow 0$. Since $L_{\ell_p}\mathcal{P}\alpha = I - (U|T|_-^{2p} - |\ell|^{2p}e^{i\omega})^{-1}(U|T|^{2p} - |\ell|^{2p}e^{i\omega})$, hence $\lim_n ||L_{\ell_p}\mathcal{P}\alpha f_n|| = 1$.

This completes the proof. ∎

Theorem 4. *Let $T = U|T| \in p$-HU and $\ell = |\ell|e^{i\omega}$.* Let *$\ell_p \in \rho(U|T|_-^{2p})$. Then $\ell \in \sigma(T)$ (or $\bar{\ell} \in \sigma(T^*)$) if and only if $0 \in \sigma(\widetilde{W_{\ell_p}})$ (respectively $0 \in \sigma((\widetilde{W_{\ell_p}})^*)$, where $\ell_p = |\ell|^{2p}e^{i\omega}$.*

Proof. From the spectral mapping theorem, $\ell \in \sigma(T)$ if and only if$\ell_p \in \sigma(U|T|^{2p})$. It holds that $\ell_p \in \sigma(U|T|^{2p})$ (or $\overline{\ell_p} \in \sigma((U|T|^{2p})^*)$ if and only if $0 \in \sigma(\widetilde{W_{\ell_p}})$ (respectively $0 \in \sigma((\widetilde{W_{\ell_p}})^*)$. So the proof is complete. ∎

For $T = U|T| \in$SHU, there exists the mosaic $\mathrm{B}_T(e^{i\theta}, r)$ of T (cf. [12, theorem V.2.5]). We define the Pincus principal function of T.

Definition 2. For $T = U|T| \in p$-HU, let $T_p = U|T|^{2p}$. Since $T_p \in$ SHU, the Pincus principal function $g_T(\cdot,\cdot)$ is defined by

$$g_T(e^{i\theta}, r) = \mathrm{Tr}\left(\mathrm{B}_{T_p}(e^{i\theta}, r^{2p})\right),$$

where Tr $(\cdot)$ is the trace on $\mathcal{D}$.

We denote the trace class of operators by $\mathcal{C}_1$.

Definition 3. For $T = U|T|$ with unitary U, T is called p-nearly normal if $[U, |T|^{2p}] = U|T|^{2p} - |T|^{2p}U \in \mathcal{C}_1$.

1-*nearly normal* is said to be *nearly normal*, simply.

Definition 4. Let $K \in \mathcal{C}_1$. The determinant of $(I - K)$ is defined by

$$\det(I - K) = \prod_{j=1}^{\nu(K)} (1 - \lambda_j(K)),$$

where we describe $\{\lambda_j(K)\}_{j=1}^{\nu(K)}$ as the sequence of non-zero-eigenvalues of K, counted according to their algebraic multiplicities [8, p. 157]. If $I - K$ is invertible, the determinant $(I - K)$ is equal to that of Xia [12, p. 175].

We denote $ABA^{-1}B^{-1}$ by $\{A, B\}$, which is called the multiplicative commutator. The following theorem is a key tool in this paper (see also [9, chapter X, lemma 2.6]).

Theorem E ([12, lemma VII.4.1]). *If $A, B \in B(\mathcal{D})$ and $[A, B] \in \mathcal{C}_1(\mathcal{D})$, then*

$$\det\left(\{e^A, e^B\}\right) = \exp\left(\mathrm{Tr}[A, B]\right).$$

Let $T = U|T| \in p$-HU. Since $T_p = U|T|^{2p} \in$SHU, we use the polar symbols of Theorem A. By [12, theorem V.2.5], for ℓ with Im$\ell > 0$ there exists uniquely $0 \le \mathrm{B}_{T_p}(e^{i\theta}, \rho) \le I$ such that

$$I + \alpha(e^{i\theta})\left(\beta(e^{i\theta}) - \ell\right)^{-1}\alpha(e^{i\theta}) = \exp\left(\int_{\sigma(|T|^{2p})} \frac{\mathrm{B}_{T_p}(e^{i\theta}, \rho)}{\rho - \ell} d\rho\right) \tag{2.2}$$

for almost all $e^{i\theta} \in \mathbf{T}$.

Theorem 5. *With the above notations, if T is p-nearly normal,*

$$\det\left(I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta})\right) = \exp\left(2p \int_{\sigma(|T|)} \frac{r^{2p-1} \cdot g_T(e^{i\theta}, r)}{r^{2p} - \ell} dr\right),$$

for almost all $e^{i\theta} \in \mathbf{T}$.

PROOF. Let $T_p = U|T|^{2p}$. Let $\log : \mathbf{C} - (-\infty, 0]i \to \mathbf{C}$ denote the branch of the logarithm normalised so that $\log i = \pi/2i$. Then using Dunford integral, by (2.2) we have

$$\log\left(I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta})\right) = \int_{\sigma(|T|^{2p})} \frac{\mathrm{B}_{T_p}(e^{i\theta}, \rho)}{\rho - \ell} d\rho$$

(see [9, p. 208]). Since T is p-nearly normal, by (2.1) the operator $\alpha(e^{i\theta})$ is compact. Hence $\alpha(e^{i\theta})\left(\beta(e^{i\theta}) - \ell\right)^{-1}\alpha(e^{i\theta})$ is compact. From this fact, if $\{e^{z_n}\}$ is the complete set of eigenvalues of $I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta})$, then $\{z_n\}$ is the complete set of eigenvalues of $\displaystyle\int_{\sigma(|T|^{2p})} \frac{\mathrm{B}_{T_p}(e^{i\theta}, \rho)}{\rho - \ell} d\rho$. Since T_p is nearly normal, by [12, theorem VII.4.2], we have

$$\det\left(I + \alpha(e^{i\theta})\left(\beta(e^{i\theta}) - \ell\right)^{-1}\alpha(e^{i\theta})\right) = \prod_n e^{z_n} = \exp\left(\sum_n z_n\right)$$

$$= \exp\left(\mathrm{Tr}\left(\int_{\sigma(|T|^{2p})} \frac{\mathrm{B}_{T_p}(e^{i\theta}, \rho)}{\rho - \ell} d\rho\right)\right).$$

Since $\mathrm{Tr}\left(\mathrm{B}_{T_p}(e^{i\theta}, \rho)\right) = g_T\left(e^{i\theta}, \rho^{\frac{1}{2p}}\right)$, we have

$$\det\left(I + \alpha(e^{i\theta})\left(\beta(e^{i\theta}) - \ell\right)^{-1}\alpha(e^{i\theta})\right) = \exp\left(\int_{\sigma(|T|^{2p})} \frac{g_T(e^{i\theta}, \rho^{\frac{1}{2p}})}{\rho - \ell} d\rho\right),$$

for almost all $e^{i\theta} \in \mathbf{T}$. Since $\rho \in \sigma(|T|^{2p})$ if and only if $\rho^{\frac{1}{2p}} \in \sigma(|T|)$, by the transformation $\rho = r^{2p}$ we have

$$\det\left(I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta})\right) = \exp\left(2p\int_{\sigma(|T|)} \frac{r^{2p-1} \cdot g_T(e^{i\theta}, r)}{r^{2p} - \ell} dr\right),$$

for almost all $e^{i\theta} \in \mathbf{T}$. So the proof is complete. ■

$\mathcal{A}_2$ denotes the set of all Laurent polynopmials of two variables r and z and has the form of $p(r, z) = \displaystyle\sum_{j=0}^{N}\sum_{k=-N}^{N} a_{jk} \cdot r^j z^k$, where N is a positive integer and a_{jk} are constant coefficients. Let X be an operator and Y be an invertible operator. For $p(r, z) = \displaystyle\sum_{j=0}^{N}\sum_{k=-N}^{N} a_{jk} \cdot r^j z^k \in \mathcal{A}_2$, we define

$$p(X, Y) = \sum_{j=0}^{N}\sum_{k=-N}^{N} a_{jk} \cdot X^j Y^k.$$

We denote the Jacobian for $p, q \in \mathcal{A}_2$ by $J(p,q)$, that is

$$J(p,q)(r,e^{i\theta}) = \frac{\partial p}{\partial r}(r,e^{i\theta}) \cdot \frac{\partial q}{\partial z}(r,e^{i\theta}) - \frac{\partial p}{\partial z}(r,e^{i\theta}) \cdot \frac{\partial q}{\partial r}(r,e^{i\theta}).$$

Then we have the following result.

Theorem F ([3, theorem 10]). *Let n be a positive integer and $T = U|T| \in \frac{1}{2n}$-HU. If T is $\frac{1}{2n}$-nearly normal, then, for $p, q \in \mathcal{A}_2$,*

$$\mathrm{Tr}\Big([p(|T|,U), q(|T|,U)] \Big) = \int\int_{\sigma(T)} J(p,q)(r,e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

Theorem 6. *Let n be a positive integer and $T = U|T| \in \frac{1}{2n}$-HU. If T is $\frac{1}{2n}$-nearly normal, then, for $p, q \in \mathcal{A}_2$,*

$$\det\Big(\{\exp(p(|T|,U)), \exp(q(|T|,U))\} \Big) = \exp\Big(\mathrm{T}r\left([p(|T|,U), q(|T|,q)]\right) \Big)$$

$$= \exp\left(\int\int_{\sigma(T)} J(p,q)(r,e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta) \right).$$

PROOF. By Theorem F, we have, for $p, q \in \mathcal{A}_2$,

$$\mathrm{Tr}\Big([p(|T|,U), q(|T|,U)] \Big) = \int\int_{\sigma(T)} J(p,q)(r,e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

Letting the exponential of both side, by Theorem E we have

$$\det\Big(\{\exp(p(|T|,U)), \exp(q(|T|,U))\} \Big) = \exp\Big(\mathrm{Tr}\left([p(|T|,U), q(|T|,q)]\right) \Big)$$

$$= \exp\left(\int\int_{\sigma(T)} J(p,q)(r,e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta) \right).$$

So the proof is complete. ■

Corollary 7. *Let n be a positive integer and $T = U|T| \in \frac{1}{2n}$-HU. If T is $\frac{1}{2n}$-nearly normal, then, for $p, q \in \mathcal{A}_2$,*

$$\det\Big(\{\exp(p(|T|,U)), \exp(q(|T|,U))\} \Big)$$

$$= \exp\left(\frac{1}{2\pi}\int\int_{\sigma(T)} J(p,q)(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)\,drd\theta\right).$$

Proof. Since $dm(\theta) = \frac{1}{2\pi}d\theta$, by Theorem 6 we have

$$\det\Big(\{\exp\left(p(|T|,U)\right), \exp\left(q(|T|,U)\right)\}\Big)$$

$$= \exp\left(\frac{1}{2\pi}\int\int_{\sigma(T)} J(p,q)(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drd\theta\right).$$

■

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References

[1] A. Aluthge, On p-hyponormal operator for $0 < p < 1$, *Integral Equations and Operator Theory* **3** (1990), 307–15.

[2] M. Chō and T. Huruya, p-hyponormal operators for $0 < p < 1/2$, *Commentationes Mathematicae/* **33** (1993), 23–9.

[3] M. Chō and T. Huruya, Trace formulae of p-hyponormal operators, *Studia Mathematica* **161** (2004), 1–18.

[4] M. Chō, T. Huruya and M. Itoh, Singular integral models for p-hyponormal operators and the Riemann-Hilbert problem, *Studia Mathematica* **130** (1998), 213–21.

[5] M. Chō, T. Huruya and M. Itoh, Riemann-Hilbert problem for characteristic functions of p-hyponormal operators, *Acta Scientiarum Mathematicarum/* (*Szeged*) **64** (1998), 271–9.

[6] M. Chō, T. Huruya and C. Li, Trace formulae associated with the polar decomposition of operators, *mathematical Proceedings of the Royal Irish Academy* **105** (2005), 57–69.

[7] M. Chō and M. Itoh, On spectra of p-hyponormal operators, *Integral Equations and Operator Theory* **23** (1995), 287–93.

[8] I.C. Goberg and M.G. Krein, Introduction to the theory of linear nonselfadjoint operators, *Translations of Mathematical Monographs/* no.18, American Mathematical Society, Providence, 1969.

[9] M. Martin and M. Putinar, *Lectures on hyponormal operators*, Birkhäuser Verlag, Basel, 1989.

[10] J.D. Pincus and D. Xia, Mosaic and principal function of hyponormal and semi-hyponormal operators, *Integral Equations and Operator Theory* **4** (1981), 134–50.

[11] D. Xia, On the non-normal operators – semi-hyponormal operators, *Sci. Sinica* **please give full title of journal 23** (1980), 700–13.

[12] D. Xia, *Spectral theory of hyponormal operators*, Birkhäuser Verlag, Basel, 1983.