

## TWO RESULTS ABOUT COMMON FIXED POINTS OF MULTIFUNCTIONS ON CONE METRIC SPACES

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### ABSTRACT

Recently, cone metric spaces have received attention because the category of cone metric spaces (even with normal constant  $M = 1$ ) is bigger than the category of metric spaces. In this paper, we shall give two results about common fixed points of two multifunctions on cone metric spaces with normal constant  $M = 1$ .

### 1. Introduction

Fixed point theory and common fixed point theory have basic roles in the applications of some branches of mathematics. The theory of common fixed points of multifunctions is a generalisation of the theory of fixed point of mappings in a sense. There are many articles about fixed points of contractive maps (for example, [3]). Also, there are many articles about fixed point theory and fixed points of multifunctions (for example, [1; 4; 6]). In some the articles, the authors define an order by using a cone in a vector space. In this manner, Huang and Zhang [2] defined cone metric spaces and were able to generalise some of the fixed point theory of metric spaces to cone metric spaces. Indeed, they substituted a ‘metric’ that has values in a normed space rather than the real line and they used the theory of normal cones in order to obtain their results. As we know, most of known cones are normal with normal constant  $M = 1$ . In this paper, we shall give two results about common fixed points of two multifunctions on the cone metric spaces with normal constant  $M = 1$ .

Let  $E$  be a real Banach space and  $P$  a subset of  $E$ .  $P$  is called a cone whenever

- (i)  $P$  is closed, non-empty and  $P \neq \{0\}$ ,
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and non-negative real numbers  $a, b$ , and
- (iii)  $P \cap (-P) = \{0\}$ .

For a given cone  $P \subseteq E$ , we can define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ .  $x < y$  will stand for  $x \leq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int}P$ , where  $\text{int}P$  denotes the interior of  $P$  ([2]).

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The cone  $P$  is called normal if there is a number  $M > 0$  such that for all  $x, y \in E$

$$0 \leq x \leq y \text{ implies } \|x\| \leq M\|y\|.$$

The least positive number satisfying the above is called the normal constant of  $P$  ([2]). It is clear that  $M \geq 1$ .

In the following, let  $E$  be a normed linear space,  $P$  be a cone in  $E$  satisfying  $\text{int}(P) \neq \emptyset$ , and ' $\leq$ ' denote the partial ordering on  $E$  with respect to  $P$ .

**Definition 1.1.** Let  $X$  be a non-empty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

(d<sub>1</sub>)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,

(d<sub>2</sub>)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , and

(d<sub>3</sub>)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$ , and  $(X, d)$  is called a cone metric space ([2]).

**Example 1.2.** Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|, \alpha|x - y|)$ , where  $\alpha \geq 0$  is a constant. Then  $P$  is a normal cone with normal constant  $M = 1$  and  $(X, d)$  is a cone metric space ([2]).

**Example 1.3.** Let  $E = \ell^1$ ,  $P = \{\{x_n\}_{n \geq 1} \in E : x_n \geq 0, \text{ for all } n\}$ ,  $(X, \rho)$  a metric space and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = \{\frac{\rho(x, y)}{2^n}\}_{n \geq 1}$ . Then  $P$  is a normal cone with normal constant  $M = 1$  and  $(X, d)$  is a cone metric space.

This example shows that the category of cone metric spaces is bigger than the "one" metric spaces.

**Definition 1.4.** ([2]) Let  $(X, d)$  be a cone metric space,  $x \in X$  and  $\{x_n\}_{n \geq 1}$  a sequence in  $X$ . Then

(i)  $\{x_n\}_{n \geq 1}$  is said to converge to  $x$  whenever for every  $c \in E$  with  $0 \ll c$  there is a natural number  $N$  such that  $d(x_n, x) \ll c$  for all  $n \geq N$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$ .

(ii)  $\{x_n\}_{n \geq 1}$  is said to be a Cauchy sequence whenever for every  $c \in E$  with  $0 \ll c$  there is a natural number  $N$  such that  $d(x_n, x_m) \ll c$  for all  $n, m \geq N$ .

(iii)  $(X, d)$  is called a complete cone metric space if every Cauchy sequence is convergent.

**Definition 1.5.** Let  $(X, d)$  be a cone metric space and  $B \subseteq X$ .

- (i) A point  $b$  in  $B$  is called an interior point of  $B$  whenever there exists a point  $p$ ,  $0 \ll p$ , such that

$$N(b, p) \subseteq B,$$

where  $N(b, p) := \{y \in X : d(y, b) \ll p\}$ .

- (ii) A subset  $A \subseteq X$  is called open if each element of  $A$  is an interior point of  $A$ .

The family  $\mathcal{B} = \{N(x, e) : x \in X, 0 \ll e\}$  is a sub-basis for a topology on  $X$ . We denote this cone topology by  $\tau_c$ . The topology  $\tau_c$  is Hausdorff and first countable ([5]).

## 2. Main results

Most of familiar cones are normal with normal constant  $M = 1$ . But for each  $k > 1$  there are cones with normal constant  $M > k$ . First we prove the following two lemmas.

**Lemma 2.1.** Let  $(X, d)$  be a cone metric space,  $P$  a normal cone with normal constant  $M = 1$  and  $A$  a compact set in  $(X, \tau_c)$ . Then, for every  $x \in X$  there exists  $a_0 \in A$  such that

$$\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|.$$

PROOF. Let  $x \in X$  be given. Define  $f_x : X \rightarrow [0, \infty)$  by  $f_x(y) = \|d(x, y)\|$ . Let  $y \in X$  and  $\varepsilon > 0$ . Choose  $0 \ll c$  such that  $\|c\| < \varepsilon$  and suppose that  $z \in N(y, c)$ . Note that, by using the normality and the relations

$$d(y, x) \leq d(y, z) + d(z, x), \quad d(z, x) \leq d(z, y) + d(y, x),$$

we have

$$\|d(y, x)\| - \|d(z, x)\| \leq \|d(y, z)\| < \varepsilon, \quad \|d(z, x)\| - \|d(y, x)\| \leq \|d(z, y)\| < \varepsilon.$$

Thus,  $|f_x(y) - f_x(z)| < \varepsilon$ . Hence,  $f_x$  is continuous on  $X$ . Since  $A$  is compact,  $f_x(A)$  is compact subset of  $[0, \infty)$ , and so there exists  $a_0 \in A$  such that

$$\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|.$$

■

**Lemma 2.2.** Let  $(X, d)$  be a cone metric space,  $P$  a normal cone with normal constant  $M = 1$  and  $A, B$  two compact sets in  $(X, \tau_c)$ . Then,

$$\sup_{x \in B} d'(x, A) < \infty,$$

where  $d'(x, A) = \inf_{a \in A} \|d(x, a)\|$ , for each  $x$  in  $X$ .

PROOF. Define  $g_A : X \rightarrow [0, \infty)$  by  $g_A(x) = \inf_{a \in A} \|d(x, a)\|$ . Let  $x \in X$  and  $\varepsilon > 0$ . Choose  $0 \ll c$  such that  $\|c\| < \varepsilon$  and suppose that  $y \in N(x, c)$ . Note that by using the normality and the relation  $d(x, a) \leq d(x, y) + d(y, a)$  we have

$$g_A(x) - g_A(y) \leq \|d(x, y)\|.$$

Thus,  $|g_A(x) - g_A(y)| < \varepsilon$ . Hence,  $g_A$  is continuous on  $X$ . Since  $B$  is compact,  $g_A(B)$  is compact subset of  $[0, \infty)$ . This completes the proof. ■

**Definition 2.1.** ([6]) Let  $(X, d)$  be a cone metric space,  $P$  a normal cone with normal constant  $M = 1$ ,  $\mathcal{H}_c(X)$  the set of all compact subsets of  $(X, \tau_c)$  and  $A \in \mathcal{H}_c(X)$ . By using Lemma 2.2, we can define

$$h_A : \mathcal{H}_c(X) \rightarrow [0, \infty) \text{ and } d_H : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow [0, \infty)$$

by

$$h_A(B) = \sup_{x \in A} d'(x, B) \text{ and } d_H(A, B) = \max\{h_A(B), h_B(A)\},$$

respectively.

**Remark 2.1.** Let  $(X, d)$  be a cone metric space with normal constant  $M = 1$ . Define  $\rho : X \times X \rightarrow [0, \infty)$  by  $\rho(x, y) = \|d(x, y)\|$ . Then,  $(X, \rho)$  is a metric space. This implies that for each  $A, B \in \mathcal{H}_c(X)$  and  $x, y \in X$ , we have the following relations

- (i)  $d'(x, A) \leq \|d(x, y)\| + d'(y, A)$ ,
- (ii)  $d'(x, A) \leq d'(x, B) + h_B(A)$ , and
- (iii)  $d'(x, A) \leq \|d(x, y)\| + d'(y, B) + h_B(A)$ .

**Theorem 2.3.** Let  $(X, d)$  be a complete cone metric space with normal constant  $M = 1$  and  $T_1, T_2 : X \rightarrow \mathcal{H}_c(X)$  two multifunctions satisfying the relation

$$d_H(T_1x, T_2y) \leq c(d'(x, T_1x) + d'(y, T_2y))$$

for all  $x, y \in X$ , where  $c \in (0, \frac{1}{2})$  is a constant. Then,  $T_1$  and  $T_2$  have a common fixed point, that is, there exists  $x \in X$  such that  $x \in T_1x$  and  $x \in T_2x$ .

PROOF. Let  $x_0 \in X$  be given. By Lemma 2.1, there is  $x_1 \in T_1x_0$  such that

$$d'(x_0, T_1x_0) = \|d(x_0, x_1)\|.$$

Also, there is  $x_2 \in T_2x_1$  such that

$$d'(x_1, T_2x_1) = \|d(x_1, x_2)\|.$$

Thus, we obtain a sequence  $\{x_n\}_{n \geq 1}$  in  $X$  such that

$$x_{2n-1} \in T_1x_{2n-2}, \quad x_{2n} \in T_2x_{2n-1},$$

$$d'(x_{2n-2}, T_1x_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|,$$

and

$$d'(x_{2n-1}, T_2x_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|,$$

for all  $n \geq 1$ . Thus, for all  $n \geq 1$  we have

$$\begin{aligned} \|d(x_{2n}, x_{2n+1})\| &= d'(x_{2n}, T_1x_{2n}) \\ &\leq h_{T_2x_{2n-1}}(T_1x_{2n}) \leq d_H(T_2x_{2n-1}, T_1x_{2n}) \\ &\leq c(d'(x_{2n}, T_1x_{2n}) + d'(x_{2n-1}, T_2x_{2n-1})) \\ &= c(\|d(x_{2n}, x_{2n+1})\| + \|d(x_{2n-1}, x_{2n})\|). \end{aligned}$$

Hence,  $\|d(x_{2n}, x_{2n+1})\| \leq \frac{c}{1-c} \|d(x_{2n-1}, x_{2n})\|$  for all  $n \geq 1$ . Also,

$$\|d(x_{2n-1}, x_{2n})\| = d'(x_{2n-1}, T_2x_{2n-1}) \leq h_{T_1x_{2n-2}}(T_2x_{2n-1})$$

$$\leq d_H(T_1x_{2n-2}, T_2x_{2n-1}) \leq c(\|d(x_{2n-2}, x_{2n-1})\| + \|d(x_{2n-1}, x_{2n})\|),$$

for all  $n \geq 1$ . Hence,

$$\|d(x_{2n-1}, x_{2n})\| \leq \frac{c}{1-c} \|d(x_{2n-2}, x_{2n-1})\|,$$

for all  $n \geq 1$ . This implies that  $\|d(x_m, x_{m+1})\| \leq \frac{c}{1-c} \|d(x_{m-1}, x_m)\|$ , for all  $m \geq 1$ . Put  $s = \frac{c}{1-c}$ . Then, for  $n > m$  we have

$$\|d(x_n, x_m)\| \leq \sum_{i=m+1}^n \|d(x_i, x_{i-1})\|$$

$$\leq (s^{n-1} + \cdots + s^m) \|d(x_0, x_1)\| \leq \frac{s^m}{1-s} \|d(x_0, x_1)\|.$$

This implies that

$$\lim_{m, n \rightarrow \infty} \|d(x_n, x_m)\| = 0.$$

By [2; lemma 4],  $\{x_n\}_{n \geq 1}$  is a Cauchy sequence in  $X$ . Thus, there exists  $x^* \in X$  such that  $x_n \rightarrow x^*$ . Now by using Remark 2.1, we have

$$\begin{aligned} d'(x^*, T_1x^*) &\leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*) \\ &\leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\ &\leq \|d(x^*, x_{2n})\| + c(d'(x_{2n-1}, T_2x_{2n-1}) + d'(x^*, T_1x^*)), \end{aligned}$$

for all  $n \geq 1$ . Hence,

$$\begin{aligned} d'(x^*, T_1x^*) &\leq \frac{c}{1-c}d'(x_{2n-1}, T_2x_{2n-1}) + \frac{1}{1-c}\|d(x^*, x_{2n})\| \\ &= \frac{c}{1-c}\|d(x_{2n-1}, x_{2n})\| + \frac{1}{1-c}\|d(x^*, x_{2n})\|, \end{aligned}$$

for all  $n \geq 1$ . Therefore,  $d'(x^*, T_1x^*) = 0$ . By Lemma 2.1,  $x^* \in T_1x^*$ . On the other hand, similarly we have

$$\begin{aligned} d'(x^*, T_2x^*) &\leq d'(x^*, T_1x_{2n}) + h_{T_1x_{2n}}(T_2x^*) \\ &\leq d'(x^*, T_1x_{2n}) + d_H(T_1x_{2n}, T_2x^*) \\ &\leq \|d(x^*, x_{2n+1})\| + c(d'(x_{2n}, T_1x_{2n}) + d'(x^*, T_2x^*)), \end{aligned}$$

for all  $n \geq 1$ . Hence,

$$\begin{aligned} d'(x^*, T_2x^*) &\leq \frac{c}{1-c}d'(x_{2n}, T_1x_{2n}) + \frac{1}{1-c}\|d(x^*, x_{2n+1})\| \\ &= \frac{c}{1-c}\|d(x_{2n+1}, x_{2n})\| + \frac{1}{1-c}\|d(x^*, x_{2n+1})\|, \end{aligned}$$

for all  $n \geq 1$ . Therefore,  $d'(x^*, T_2x^*) = 0$ . By Lemma 2.1,  $x^* \in T_2x^*$ . Therefore,  $x^*$  is a common fixed point of  $T_1$  and  $T_2$ . ■

**Theorem 2.4.** *Let  $(X, d)$  be a complete cone metric space with normal constant  $M = 1$  and  $T_1, T_2 : X \rightarrow \mathcal{H}_c(X)$  two multifunctions satisfy the relation*

$$d_H(T_1x, T_2y) \leq c(d'(y, T_1x) + d'(x, T_2y))$$

for all  $x, y \in X$ , where  $c \in (0, \frac{1}{2})$  is a constant. Then,  $T_1$  and  $T_2$  have a common fixed point.

PROOF. A similar argument to that of the proof of Theorem 2.3 shows that there exists a Cauchy sequence  $\{x_n\}_{n \geq 1}$  in  $X$  such that

$$x_{2n-1} \in T_1x_{2n-2}, \quad x_{2n} \in T_2x_{2n-1},$$

$$d'(x_{2n-2}, T_1x_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|,$$

and

$$d'(x_{2n-1}, T_2x_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|,$$

for all  $n \geq 1$ . Thus, there exists  $x^* \in X$  such that  $x_n \rightarrow x^*$ . Now by using Remark 2.1, we have

$$d'(x^*, T_1x^*) \leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*)$$

$$\begin{aligned} &\leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\ &\leq \|d(x^*, x_{2n})\| + c(d'(x^*, T_2x_{2n-1}) + d'(x_{2n-1}, T_1x^*)) \\ &\leq \|d(x^*, x_{2n})\| + c\|d(x^*, x_{2n})\| + cd'(x^*, T_1x^*) + c\|d(x^*, x_{2n-1})\|, \end{aligned}$$

for all  $n \geq 1$ . Hence,

$$d'(x^*, T_1x^*) \leq \frac{1+c}{1-c}\|d(x^*, x_{2n})\| + \frac{c}{1-c}\|d(x^*, x_{2n-1})\|,$$

for all  $n \geq 1$ . Therefore,  $d'(x^*, T_1x^*) = 0$ . By Lemma 2.1,  $x^* \in T_1x^*$ . Also, we have

$$\begin{aligned} d'(x^*, T_2x^*) &\leq d'(x^*, T_1x_{2n}) + h_{T_1x_{2n}}(T_2x^*) \\ &\leq d'(x^*, T_1x_{2n}) + d_H(T_1x_{2n}, T_2x^*) \\ &\leq \|d(x^*, x_{2n+1})\| + c(d'(x^*, T_1x_{2n}) + d'(x_{2n}, T_2x^*)) \\ &\leq \|d(x^*, x_{2n+1})\| + c\|d(x^*, x_{2n+1})\| + cd'(x^*, T_2x^*) + c\|d(x^*, x_{2n})\|, \end{aligned}$$

for all  $n \geq 1$ . Hence,

$$d'(x^*, T_2x^*) \leq \frac{1+c}{1-c}\|d(x^*, x_{2n+1})\| + \frac{c}{1-c}\|d(x^*, x_{2n})\|,$$

for all  $n \geq 1$ . Therefore,  $d'(x^*, T_2x^*) = 0$ . By Lemma 2.1,  $x^* \in T_2x^*$ . Therefore,  $x^*$  is a common fixed point of  $T_1$  and  $T_2$ . ■

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