

# ON THE AREA FUNCTIONAL OF THE SECOND FUNDAMENTAL FORM OF OVALOIDS

STEVEN VERPOORT  
Katholieke Universiteit Leuven, Belgium.

[Accepted 2 June 2009. Published 11 September 2009.]

## ABSTRACT

The expression for the variation of the area functional of the second fundamental form of a hypersurface in a Euclidean space involves the so-called ‘*mean curvature of the second fundamental form*.’ Several new characteristic properties of (hyper)spheres in which the mean curvature of the second fundamental form occurs are given. In particular, it is shown that the spheres are the only ovaloids that are a critical point of the area functional of the second fundamental form under various constraints.

## 1. Introduction

The intention of this article is to contribute to the theory of hypersurfaces in a Euclidean space, of which the second fundamental form is positive-definite and accordingly can be seen as an abstract Riemannian metric.

### 1.1. *Some results on the geometry of the second fundamental form.*

The *intrinsic* geometry of this abstract metric and its influence on the shape of the original hypersurface have already been the object of extensive study. For example, it has been proved by R. Schneider that the hyperspheres are the only such compact hypersurfaces of which the second fundamental form has constant Riemannian curvature [20]. In extension, it has been demonstrated that only hyperspheres can satisfy certain further relations between the sectional curvature of the second fundamental form and the curvatures of the original compact hypersurface (e.g. [1; 8; 11; 12; 22; 24]).

The notion of mean curvature, which belongs to the *extrinsic* geometry of the hypersurface and can be characterised as a measure for the rate of area growth under deformations of the hypersurface, can be tailored to the geometry of the second fundamental form: a function that measures the rate at which the total area of the hypersurface, as surveyed in the geometry of its second fundamental form, changes under a deformation is called *the mean curvature of the second fundamental form*. This concept has been introduced by E. Glässner for surfaces in  $\mathbb{E}^3$  [5; 6] (see also, e.g., [16] for hypersurfaces in  $\mathbb{E}^{m+1}$ ).

---

doi:10.3318/PRIA.2009.109.2.187

Cite as follows: Steven Verpoort, On the area functional of the second fundamental form of ovaloids, *Mathematical Proceedings of the Royal Irish Academy* **109A** (2009), 187–200; doi:10.3318/PRIA.2009.109.2.187.

*Mathematical Proceedings of the Royal Irish Academy*, **109A** (2), 187–200 (2009) © Royal Irish Academy

### 1.2. Overview

The notation that will be adopted is outlined in §2 below. In §3, the expression for the variation of the area of the second fundamental form is given and some useful remarks regarding the mean curvature of the second fundamental form will be made.

As will be mentioned in the beginning of §4, several characterisations of Euclidean hyperspheres in which the mean curvature of the second fundamental form occurs have been found already. Two similar new results are contained in this section.

In §5 an integral formula will be derived that is helpful in §6. Here it is shown that the spheres are the only ovaloids that are a critical point of the area functional of the second fundamental form under various constraints (Theorems 10, 17 and 21).

## 2. Notation

For a manifold  $M$ , the collection of all vector fields on  $M$  will be denoted by  $\mathfrak{X}(M)$  whereas  $\mathfrak{F}(M)$  stands for the collection of all real-valued functions on  $M$ .

If  $f : M \rightarrow \mathbb{R}$  is a real-valued function on a Riemannian manifold  $(M, g)$  with Levi-Civita connection  $\nabla$ , the *Hessian operator* of  $f$  is defined as

$$\text{Hs}_f : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : V \mapsto \nabla_V (\text{grad} f) .$$

The *Hessian* of  $f$  (notation  $\text{Hess}_f$ ) is the  $(0, 2)$ -tensor that is metrically equivalent to  $\text{Hs}_f$ . The *Laplacian* of  $f$  is defined as  $\Delta f = \text{trace} (\text{Hs}_f)$ .

For a hypersurface  $M \subseteq \mathbb{E}^{m+1}$ , the *first fundamental form* will be denoted by  $I$  or  $g$ , and is the restriction of the Euclidean scalar product  $\langle \cdot, \cdot \rangle$  to the tangent spaces of  $M$ . The *shape operator*  $A$  is defined with respect to a unit normal vector field  $N$  by

$$A : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : V \mapsto A(V) = -\bar{D}_V N ,$$

where  $\bar{D}$  is the standard connection on  $\mathbb{E}^{m+1}$ . The *second fundamental form* is given by  $\mathbb{I}(V, W) = \langle A(V), W \rangle$  for  $V, W \in \mathfrak{X}(M)$ , and the *mean curvature* is equal to  $H = \frac{1}{m} \text{trace} A$ . The *Gaussian curvature* of a surface in  $\mathbb{E}^3$  is denoted by  $K$ .

A hypersurface  $M \subseteq \mathbb{E}^{m+1}$  will be called *locally strongly convex* if the second fundamental form  $\mathbb{I}$  is positive-definite. A *hyperovaloid* (resp. *ovaloid*) is a compact, locally strongly convex hypersurface  $M \subseteq \mathbb{E}^{m+1}$  (resp.  $\mathbb{E}^3$ ).

The second fundamental form furnishes an abstract Riemannian metric on a locally strongly convex hypersurface  $M$ , and an index  $\mathbb{I}$  will indicate that a geometrical object is defined with respect to  $(M, \mathbb{I})$ . It should be noted that we need to choose the *interior* unit normal vector field on a hyperovaloid in order to make the second fundamental form positive-definite.

## 3. The variation of the area of the second fundamental form

The letter  $\mathcal{E}$  will designate the set of all locally strongly convex hypersurfaces in  $\mathbb{E}^{m+1}$ . Our first concern is the infinitesimal behaviour of the area functional asso-

ciated to the second fundamental form:

$$\text{Area}_{\mathbb{I}\!I} : \mathcal{E} \rightarrow \mathbb{R} : M \mapsto \text{Area}_{\mathbb{I}\!I}(M) = \int_M d\Omega_{\mathbb{I}\!I} = \int_M \sqrt{\det A} \, d\Omega.$$

By a *deformation* of a hypersurface  $M \subseteq \mathbb{E}^{m+1}$  will be understood a smooth mapping (for some  $\varepsilon > 0$ )

$$\mu : ]-\varepsilon, \varepsilon[ \times M \rightarrow \mathbb{E}^{m+1} : (t, n) \mapsto \mu_t(n),$$

such that for some compact set  $\mathcal{M} \subseteq M$

$$\begin{cases} \text{for all } n \in M, & \mu_0(n) = n; \\ \text{for all } n \in M \setminus \mathcal{M} \text{ and all } t \in ]-\varepsilon, \varepsilon[, & \mu_t(n) = n. \end{cases}$$

The  $\mathbb{E}^{m+1}$ -valued vector field sending  $n \in M$  to  $\left. \frac{\partial}{\partial t} \right|_{t=0} \mu_t(n)$  will be called the *deformation vector field*.

The proof of the following theorem, which was given by E. Glässner in [5] in dimension  $m = 2$  and in [29] for  $m \geq 2$ , has been omitted here.

**Theorem 1** (E. Glässner). *Let  $M$  be a locally strongly convex hypersurface in  $\mathbb{E}^{m+1}$ . The variation of the area functional of the second fundamental form along a deformation  $\mu$  with deformation vector field  $X$  is given by*

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \text{Area}_{\mathbb{I}\!I}(\mu_t(M)) = \int_M \left\{ \frac{m}{2} H + \frac{1}{4} \Delta_{\mathbb{I}\!I}(\log \det A) \right\} \langle -N, X \rangle \, d\Omega_{\mathbb{I}\!I}.$$

**Definition 2.** Let  $M$  be a locally strongly convex hypersurface in  $\mathbb{E}^{m+1}$ . The *mean curvature of the second fundamental form*  $H_{\mathbb{I}\!I}$  is defined by

$$H_{\mathbb{I}\!I} = \frac{m}{2} H + \frac{1}{4} \Delta_{\mathbb{I}\!I}(\log \det A).$$

If  $H_{\mathbb{I}\!I} = 0$ , the surface will be called  *$\mathbb{I}\!I$ -minimal*.

**Remark 3.**  $\mathbb{I}\!I$ -minimal hyperovaloids do not exist in  $\mathbb{E}^{m+1}$ .

**Remarks 4.** This terminology was introduced by E. Glässner [5; 6] for surfaces in  $\mathbb{E}^3$  (see also, e.g., [16] for hypersurfaces in  $\mathbb{E}^{m+1}$ ). There is some inaccuracy in the terminology, since a critical point of  $\text{Area}_{\mathbb{I}\!I}$  is not necessarily a minimum: for  $m = 2$ , M. Wiehe [30, remark 4.4] has shown that the second-order variation of  $\text{Area}_{\mathbb{I}\!I}$  is *negative-definite* for locally strongly convex surfaces in  $\mathbb{E}^3$  with  $H_{\mathbb{I}\!I} = 0$ . Despite what might be suggested by the name *mean curvature of the second fundamental form*, the quantity  $H_{\mathbb{I}\!I}$  is not determined by the second fundamental form  $\mathbb{I}\!I$  solely. This is, of course, in contrast with the *scalar curvature of the second fundamental*

form, which will be denoted by  $S_{\mathbb{I}}$  and is given by [20]

$$S_{\mathbb{I}} = m(m-1)H + \mathcal{P} - \frac{\mathbb{I}(\text{grad}_{\mathbb{I}}\det A, \text{grad}_{\mathbb{I}}\det A)}{4(\det A)^2} \quad (3.1)$$

for a non-negative function  $\mathcal{P}$ . If  $M$  is an ovaloid in three-dimensional Euclidean space, we will denote the Gaussian curvature of  $(M, \mathbb{I})$  as  $K_{\mathbb{I}} = \frac{1}{2}S_{\mathbb{I}}$ .

**Remark 5.** F. Manhart [16] has introduced the so-called  $\mathbb{I}$ -normal field

$$N_{\mathbb{I}} = \sqrt{\det A} N - \text{grad}_{\mathbb{I}} \sqrt{\det A}$$

along a hyperovaloid  $M \subseteq \mathbb{E}^{m+1}$ . Adopting the language of [23], this is a relative normal vector field with relative shape operator

$$A_{(N_{\mathbb{I}})} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : V \mapsto A_{(N_{\mathbb{I}})}(V) = -\bar{D}_V N_{\mathbb{I}},$$

the trace of which is given by  $2\sqrt{\det A} H_{\mathbb{I}}$ .

#### 4. Characterisations of Euclidean hyperspheres

Several characterisations of the Euclidean hyperspheres in which the function  $H_{\mathbb{I}}$  occurs have been found already. For example, it is known that the hyperspheres are the only hyperovaloids in  $\mathbb{E}^{m+1}$  (with  $m \geq 2$ ) that satisfy  $2(m-1)H_{\mathbb{I}} \geq S_{\mathbb{I}}$  (see [17, Satz 3.7, with  $\alpha = \frac{1}{2}$ ]). Furthermore, an ovaloid in  $\mathbb{E}^3$  for which any of the functions  $K_{\mathbb{I}} - \sqrt{K}$ ,  $H_{\mathbb{I}} - H$  and  $H_{\mathbb{I}} - K_{\mathbb{I}}$  do not change sign is a sphere (see [12; 25; 27, Korollar 1, {(b), with  $\alpha = 0$ } along with {(c), with  $\alpha = \frac{1}{2}$ }]).

The two following theorems, the first one of which generalises results of [25], give similar characterisations of hyperspheres.

**Theorem 6.** *If a hyperovaloid  $M \subseteq \mathbb{E}^{m+1}$  satisfies either  $H_{\mathbb{I}} = \frac{m}{2}H + f(\det A)$  or  $H_{\mathbb{I}} = Hf(\det A)$  for an increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , then  $M$  is a hypersphere.*

**PROOF OF THEOREM 6.** Let  $p_+$  and  $p_-$  be two points where  $\det A$  attains its maximum and minimum, respectively. In the first case, we necessarily have

$$\begin{aligned} 4f(\det A)|_{p_+} &= \Delta_{\mathbb{I}}(\log \det A)|_{p_+} \leq 0 \leq \Delta_{\mathbb{I}}(\log \det A)|_{p_-} \\ &= 4f(\det A)|_{p_-} \leq 4f(\det A)|_{p_+}. \end{aligned}$$

This shows that  $\det A$  is constant, which is only possible if  $M$  is a hypersphere [28].

In the second case,

$$(Hf(\det A))|_{p_+} = H_{\mathbb{I}}(p_+) = \left( \frac{m}{2}H + \frac{1}{4}\Delta_{\mathbb{I}}(\log \det A) \right) \Big|_{p_+} \leq \frac{m}{2}H(p_+)$$

and consequently, every  $p \in M$  satisfies

$$f(\det A)|_p \leq f(\det A)|_{p_+} \leq \frac{m}{2}.$$

But this means  $\Delta_{\mathbb{I}}(\log \det A) \leq 0$ , and  $M$  is a hypersphere. ■

**Theorem 7.** *If  $M \subseteq \mathbb{E}^{m+1}$  is a hyperovaloid, then there holds*

$$\int H_{\mathbb{I}} \, d\Omega \geq \frac{m}{2} \int H \, d\Omega$$

with equality if and only if  $M$  is a hypersphere.

PROOF OF THEOREM 7. This is an immediate consequence of the relation

$$\Delta_{\mathbb{I}}(\log \det A) = \operatorname{div}(\operatorname{grad}_{\mathbb{I}}(\log \det A)) + \frac{1}{2} \mathbb{I}(\operatorname{grad}_{\mathbb{I}}(\log \det A), \operatorname{grad}_{\mathbb{I}}(\log \det A))$$

that can easily be obtained by direct computation. ■

### 5. An integral formula

It is possible to adapt an integral formula of H. Minkowski to the geometry of the second fundamental form. By  $P$  shall be denoted the position vector field of  $\mathbb{E}^{m+1}$  with respect to an arbitrary origin, and  $\rho$  will stand for the support function

$$\rho \equiv \langle -N, P \rangle : M \rightarrow \mathbb{R}.$$

**Theorem 8.** *The following integral formula holds on any hyperovaloid  $M \subseteq \mathbb{E}^{m+1}$ :*

$$\frac{m}{2} \operatorname{Area}_{\mathbb{I}}(M) = \int_M H_{\mathbb{I}} \rho \, d\Omega_{\mathbb{I}}. \quad (5.1)$$

PROOF OF THEOREM 8. By deforming  $M$  in the direction of the position vector field, we obtain the hypersurface

$$M_t = \{n + tP_{(n)} \mid n \in M\} \quad (\text{with } t \in ]-1, +\infty[),$$

which is a rescaling of  $M$  with a factor  $1 + t$ . Such a homothety magnifies the Gauss-Kronecker curvature  $\det A$  with a factor  $\left(\frac{1}{1+t}\right)^m$ , whereas  $m$ -dimensional areas are multiplied with a factor  $(1 + t)^m$ . Hence there holds  $\operatorname{Area}_{\mathbb{I}}(M_t) = (1 + t)^{m/2} \operatorname{Area}_{\mathbb{I}}(M)$  and consequently,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \operatorname{Area}_{\mathbb{I}}(M_s) = \frac{m}{2} \operatorname{Area}_{\mathbb{I}}(M).$$

If, on the other hand,  $\{M_t\}$  is considered as a deformation of  $M$  with deformation vector field  $P$ , the definition of the mean curvature of the second fundamental form

gives us

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \text{Area}_{\mathbb{I}}(M_t) = \int_M H_{\mathbb{I}} \langle -N, P \rangle d\Omega_{\mathbb{I}}.$$

The integral formula is a consequence of these two last equalities. ■

**Remark 9.** It is interesting to notice that two integral formulae that appear in Minkowski's article ([19]) have been obtained already by J.H. Jellett ([10]) (see also [3]). The support function plays an important role in these formulae, and the fact that this function has been described already in W.R. Hamilton's article ([7]), suggests that it might have been Hamilton's influence that made the other Dublin mathematician find the formulae. Extensions of these integral formulae can be found in [9; 15], for example, and the integral formula (5.1) may be seen as a special case of the latter one. The integral formula (5.1) can also be obtained by specialising the generalisation of Minkowski's integral formula within the framework of relative differential geometry (see [21; 28] for example) to Manhart's  $\mathbb{I}$ -normal field.

## 6. Variational characterisations of Euclidean spheres

In this section, I show that the spheres are the only critical points of the area functional of the second fundamental form under three different constraints.

### 6.1. First variational problem

The proof of our first variational characterisation of the sphere (Theorem 10, below) makes use of the Jellett–Minkowski integral formulae.

**Theorem 10.** *If an ovaloid in  $\mathbb{E}^3$  is a critical point of the the  $\mathbb{I}$ -area with respect to variations under which the integral mean curvature  $\int H d\Omega$  is preserved, then it is a sphere.*

PROOF OF THEOREM 10. Under a deformation  $\mu$  of an ovaloid  $M \subseteq \mathbb{E}^3$  with variational vector field  $X$ , there holds

$$\left\{ \begin{array}{l} \left. \frac{\partial}{\partial t} \right|_{t=0} \int_{\mu_t(M)} H d\Omega = \int_M K \langle -N, X \rangle d\Omega; \\ \left. \frac{\partial}{\partial t} \right|_{t=0} \text{Area}_{\mathbb{I}}(\mu_t(M)) = \int_M H_{\mathbb{I}} \sqrt{K} \langle -N, X \rangle d\Omega. \end{array} \right.$$

Now assume an ovaloid  $M$  is a critical point of the area functional of the second fundamental form, under the constraint that the integral mean curvature be preserved. The Euler-Lagrange equation, which is satisfied on  $M$ , reads  $H_{\mathbb{I}} = C \sqrt{K}$  (for  $C \in \mathbb{R}$ ). It should be observed that the constant  $C$  is necessarily greater or

equal than one:<sup>1</sup>

$$C \int \sqrt{K} \, d\Omega_{\mathbb{I}} = \int H_{\mathbb{I}} \, d\Omega_{\mathbb{I}} = \int H \, d\Omega_{\mathbb{I}} \geq \int \sqrt{K} \, d\Omega_{\mathbb{I}}.$$

On the other hand, an application of formula (5.1) and of another Jellett–Minkowski integral formula gives

$$\begin{aligned} \int \sqrt{K} \, d\Omega &= \text{Area}_{\mathbb{I}} = \int H_{\mathbb{I}} \rho \, d\Omega_{\mathbb{I}} = C \int K \rho \, d\Omega \\ &= C \int H \, d\Omega \geq \int H \, d\Omega \geq \int \sqrt{K} \, d\Omega. \end{aligned}$$

This is only possible if  $M$  is a sphere. ■

**Remark 11.** As follows from the inequality

$$\int H \, d\Omega \geq \int \sqrt{K} \, d\Omega = \text{Area}_{\mathbb{I}},$$

the spheres are actually a maximum of the functional  $\text{Area}_{\mathbb{I}}$  that has been constrained to the class consisting of all ovaloids for which the integral mean curvature has some particular value.

**Remark 12.** As an immediate consequence of Theorem 6, the spheres are the only ovaloids in  $\mathbb{E}^3$  satisfying  $H_{\mathbb{I}} = CH$  for some constant  $C$ . The following corollary of Theorem 10 is of a similar nature:

**Corollary 13.** *The spheres are the only ovaloids in  $\mathbb{E}^3$  which satisfy  $H_{\mathbb{I}} = C\sqrt{K}$  for some  $C \in \mathbb{R}$ .*

**Remark 14.** Theorem 10 can be seen as a modification of the following classical theorem: ‘If an ovaloid in  $\mathbb{E}^3$  is a critical point of the (classical) area with respect to variations under which the integral mean curvature  $\int H \, d\Omega$  is preserved, then it is a sphere.’ Namely, the critical points satisfy  $H = CK$ , and by a theorem of E.B. Christoffel ([4, p. 163]) only the spheres satisfy this equation. We shall give a proof of another modification of Christoffel’s result (Theorem 15, below), which also follows from theorem 1.d of G. Stamou [26] under the additional assumption  $K_{\mathbb{I}} > 0$ .

**Theorem 15.** *The spheres are the only ovaloids  $M \subseteq \mathbb{E}^3$  such that*

$$H_{\mathbb{I}} = CK_{\mathbb{I}} \tag{6.1}$$

for some  $C \in \mathbb{R}$ .

<sup>1</sup>This fact can also be deduced from Theorem 7.

PROOF OF THEOREM 15. Let  $M$  be an ovaloid such that  $H_{\mathbb{I}} = C K_{\mathbb{I}}$ . It should be noticed that the constant that occurs in (6.1) satisfies  $C \geq 1$ . This inequality follows from

$$\begin{aligned} 4\pi C &= \int C K_{\mathbb{I}} d\Omega_{\mathbb{I}} = \int H_{\mathbb{I}} d\Omega_{\mathbb{I}} \\ &= \int H d\Omega_{\mathbb{I}} = \int H \sqrt{K} d\Omega \geq \int K d\Omega = 4\pi. \end{aligned} \quad (6.2)$$

The following inequalities are valid at a point  $p$  where  $K$  achieves its maximal value:

$$\begin{aligned} 0 &\geq \left( \frac{1}{4} \Delta_{\mathbb{I}} \log K \right)_{(p)} = \left( H_{\mathbb{I}} - H \right)_{(p)} = \left( C K_{\mathbb{I}} - H \right)_{(p)} \\ &= \left( (C-1)H + \frac{C}{2} \mathcal{P} \right)_{(p)} \geq (C-1)H(p). \end{aligned}$$

This is only possible if  $C = 1$ , such that equality occurs in (6.2). This is only possible if  $H = \sqrt{K}$  throughout on  $M$ . ■

**Remark 16.** For an adaption of Corollary 13 for curves in the plane, see [29, p. 132]. The corresponding result can be reworded as follows: ‘If a simple closed plane curve with strictly positive curvature is a critical point of the length functional of the second fundamental form ( $= \int \sqrt{\kappa} ds$ ) with respect to deformations under which the (classical) length is preserved, then it is a circle.’ This leads us towards a second variational problem.

### 6.2. Second variational problem.

A basic ingredient in the proof of the following theorem is the differential equation

$$\text{Hess}_{\psi} = \frac{\Delta\psi}{m} g$$

for a function  $\psi \in \mathfrak{F}(M)$  on an  $m$ -dimensional Riemannian manifold  $(M, g)$ .

The above differential equation naturally arises in the context of conformal transformations between Einstein spaces, between space forms, and concircular transformations (see [13, lemma 2, proposition 3 and proposition 8, respectively]) and was studied initially by H.W. Brinkmann, A. Fialkow, Y. Tashiro, K. Yano *et al*; we refer to W. Kühnel’s survey text [13].

**Theorem 17.** *If an ovaloid in  $\mathbb{E}^3$  is a critical point of the  $\mathbb{I}$ -area with respect to deformations under which the (classical) area is preserved, then it is a sphere.*

PROOF OF THEOREM 17. An ovaloid is a critical point of this variational problem if and only if the following relation is satisfied for some  $C \in \mathbb{R}$ :

$$H_{\mathbb{I}} \sqrt{K} = CH. \quad (6.3)$$

Let us temporarily define an *abstract ovaloid* as a compact two-dimensional

Riemannian manifold  $(M, g)$  such that  $M$  is diffeomorphic to a sphere and  $g$  has strictly positive Gaussian curvature. The notation  $\mathcal{E}_{\text{ABS}} = \{ \text{abstract ovaloids} \}$  will be adopted.

It should be remarked that an abstract ovaloid is actually not so abstract, since the Weyl embedding problem has been solved by H. Weyl, H. Lewy, *et al.* More precisely, for every abstract ovaloid  $(M_{\text{ABS}}, g)$  an ovaloid  $M \subseteq \mathbb{E}^3$  can be found that is isometric with  $(M_{\text{ABS}}, g)$ . Moreover, the congruence theorem of S. Cohn-Vossen gives us that  $M$  is unique up to a congruence of  $\mathbb{E}^3$ .

For any deformation  $g^{(t)}$  of the metric of an abstract ovaloid  $(M, g)$ , as well  $(M, g^{(t)})$  is an abstract ovaloid for sufficiently small  $|t|$ , and hence variational problems can be posed on the class  $\mathcal{E}_{\text{ABS}}$ . Because of the previous remark, this is essentially not different from the study of variational problems on the class of ‘*real*’ ovaloids.

We introduce the following functionals:

$$\left\{ \begin{array}{l} \mathcal{F} : \mathcal{E}_{\text{ABS}} \rightarrow \mathbb{R} : (M, g) \mapsto \int_M d\Omega; \\ \mathcal{F}_{\text{II}} : \mathcal{E}_{\text{ABS}} \rightarrow \mathbb{R} : (M, g) \mapsto \int_M \sqrt{K} d\Omega. \end{array} \right.$$

Of course,  $K$  stands for the Gaussian curvature of the metric  $g$  on  $M$ , and  $d\Omega$  is the area element of this abstract metric.

If a deformation  $g^{(t)} = g + th + \mathcal{O}(t^2)$  of the metric of an abstract ovaloid  $(M, g)$  has been given, then there exists a unique operator  $\sigma$  that is symmetric with respect to the metric  $g$  and satisfies  $h(V, W) = g(\sigma(V), W)$  for all  $V, W \in \mathfrak{X}(M)$ . The following variational formulae can be verified (see [29, pp. 66 ff.], for example, for some details):

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \Big|_{t=0} \mathcal{F}(M, g^{(t)}) = \frac{1}{2} \int_M \text{trace } \sigma d\Omega; \\ \frac{\partial}{\partial t} \Big|_{t=0} \mathcal{F}_{\text{II}}(M, g^{(t)}) = \frac{1}{4} \int_M \text{trace} \left\{ \left( \left( \sqrt{K} - \Delta \left( \frac{1}{\sqrt{K}} \right) \right) \text{id} + \text{Hs}_{\frac{1}{\sqrt{K}}} \right) \circ \sigma \right\} d\Omega. \end{array} \right.$$

By an application of the lemma that is stated in [2, p. 168], it can be concluded that an abstract ovaloid is a critical point for  $\mathcal{F}_{\text{II}}$  under the constraint that  $\mathcal{F}$  be preserved if and only if the following operator-valued equation is satisfied:

$$\left( \sqrt{K} - \Delta \left( \frac{1}{\sqrt{K}} \right) \right) \text{id} + \text{Hs}_{\frac{1}{\sqrt{K}}} = C \text{id} \quad (C \in \mathbb{R}). \quad (6.4)$$

Let now a ‘*real*’ and non-spherical ovaloid  $M \subseteq \mathbb{E}^3$  be given. Assume  $M$  is a critical point of the II-area with respect to deformations under which the (classical) area be preserved. Then the abstract ovaloid  $(M, \text{I})$  is a critical point of  $\mathcal{F}_{\text{II}}$  under the constraint that  $\mathcal{F}$  be preserved, and consequently (6.4) is satisfied.

(If the trace of the composition of both sides of the above equation with the

shape operator is taken, there results (6.3), which particularly shows that the constants  $C$  occurring in (6.3) and (6.4) are equal.)

If the trace of both sides of the above equation (6.4) is taken, there results

$$2 \left( \frac{1}{\psi} - C \right) = \Delta \psi, \quad (6.5)$$

where  $\psi$  stands for the function  $\psi = \frac{1}{\sqrt{K}} \in \mathfrak{F}(M)$ .

It follows from the above equation (6.4) that

$$\text{Hess}_\psi = \frac{\Delta \psi}{2} g. \quad (6.6)$$

Let  $p_+$  be a point where  $K$  attains its global maximum (and hence  $\psi$  its global minimum). Similarly the global minimum of  $K$  (and hence the global maximum of  $\psi$ ) is attained in a point  $p_-$  of  $M$ . The notation

$$k_+ = \sqrt{K} \Big|_{p_+} = \frac{1}{\psi} \Big|_{p_+} \quad \text{and} \quad k_- = \sqrt{K} \Big|_{p_-} = \frac{1}{\psi} \Big|_{p_-}$$

will be adopted. In the remainder of this proof it will be shown that the assumption that  $M$  is non-spherical, which is equivalent to the inequality

$$k_- < k_+, \quad (6.7)$$

allows a contradiction to be deduced.

According to [13, theorem 21],  $p_+$  and  $p_-$  are the only critical points of  $\psi$  on  $M$ . Moreover, according to lemma 22 of [13],  $M \setminus \{p_-, p_+\}$  (with the first fundamental form metric) is isometric to a warped product  $]u_0, u_1[ \times S^1(\ell)$ , the slices of which correspond to the level sets of the function  $\psi$ . Thus the function  $\psi$  can be seen as a function of the first factor only, *i.e.*,  $\psi : ]u_0, u_1[ \rightarrow \mathbb{R}$ , and, according to [13], the derivative of  $\psi$  is exactly the warping function. Thus if  $S^1(\ell)$  is described with a co-ordinate  $v \in ]0, 2\pi\ell[$ , there holds

$$M \setminus \{p_-, p_+\} = ]u_0, u_1[ \times_{\psi'} ]0, 2\pi\ell[,$$

with the end-points of the second interval identified.

Then  $M \setminus \{p_-, p_+\}$  can be described with co-ordinates  $(u, v)$  and the (first fundamental form) metric is, more precisely,

$$g = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (\psi'(u))^2 \end{pmatrix} = du^2 + (\psi'(u))^2 dv^2. \quad (6.8)$$

After possibly having reflected the  $u$ -interval, the function  $\psi : ]u_0, u_1[ \rightarrow \mathbb{R}$  is strictly increasing, and there holds

$$\psi(u_0) := \lim_{u \rightarrow u_0} \psi(u) = \frac{1}{\sqrt{K}} \Big|_{p_+} = \frac{1}{k_+} \quad \text{and} \quad \psi(u_1) := \lim_{u \rightarrow u_1} \psi(u) = \frac{1}{\sqrt{K}} \Big|_{p_-} = \frac{1}{k_-}.$$

The expression (6.8) for the metric implies that  $\frac{\Delta\psi}{2} = \psi''$ . Together with (6.5), this yields the following ordinary differential equation for  $\psi$ :

$$\psi'' = \frac{1}{\psi} - C.$$

In particular, since  $\lim_{u \rightarrow u_0} \psi(u)$  is a well-defined number in  $\mathbb{R}_0^+$ ,

$$\lim_{u \rightarrow u_0} \psi''(u) = k_+ - C \quad (6.9)$$

is also a well-defined real number (which, of course, is denoted simply by  $\psi''(u_0)$ ).

Now three global geometric invariants can be calculated with respect to this explicit description of the metric on  $M$ . First of all, the total Gaussian curvature can be expressed as

$$4\pi = \int K \, d\Omega = \int_{u_0}^{u_1} \int_0^{2\pi\ell} \frac{\psi'(u)}{(\psi(u))^2} \, dv \, du = 2\pi\ell(k_+ - k_-). \quad (6.10)$$

Secondly, the  $\mathbb{I}$ -area of  $M$  satisfies

$$\text{Area}_{\mathbb{I}}(M) = \int \sqrt{K} \, d\Omega = \int_{u_0}^{u_1} \int_0^{2\pi\ell} \frac{\psi'(u)}{\psi(u)} \, dv \, du = 2\pi\ell \ln\left(\frac{k_+}{k_-}\right). \quad (6.11)$$

Thirdly, the area of  $M$  satisfies

$$\text{Area}(M) = \int d\Omega = \int_{u_0}^{u_1} \int_0^{2\pi\ell} \psi'(u) \, dv \, du = 2\pi\ell \frac{(k_+ - k_-)}{k_+ k_-}. \quad (6.12)$$

As immediately follows from a combination of (6.3) together with two Jellett–Minkowski integral equations, these last two quantities are related by

$$\text{Area}_{\mathbb{I}}(M) = \int H_{\mathbb{I}} \rho \, d\Omega_{\mathbb{I}} = \int H_{\mathbb{I}} \sqrt{K} \rho \, d\Omega = C \int H \rho \, d\Omega = C \text{Area}(M). \quad (6.13)$$

Now it will be shown that the constant  $C$  satisfies

$$C = \frac{(k_+ + k_-)}{2}. \quad (6.14)$$

This can be concluded by a calculation of the length of the geodesic circle  $\gamma$  of radius  $\varepsilon$  and centre  $p_+$  in two different ways. Firstly, the curve  $\gamma$  is described in the co-ordinate system  $(u, v)$  by the equation  $u = u_0 + \varepsilon$ , and it can be seen that

$$\text{Length}(\gamma) = \int_0^{2\pi\ell} \psi'(u_0 + \varepsilon) \, dv = 2\pi\ell \psi'(u_0 + \varepsilon) = 2\pi\ell \psi''(u_0) \varepsilon + \mathcal{O}(\varepsilon^2).$$

Secondly, since  $\gamma$  is a geodesic circle of radius  $\varepsilon$ ,

$$\text{Length}(\gamma) = 2\pi\varepsilon + \mathcal{O}(\varepsilon^2).$$

A comparison of the two last expressions gives that

$$\psi''(u_0)\ell = 1. \quad (6.15)$$

Consequently, making use respectively of (6.10), (6.15) and (6.9), it can be seen that

$$\ell(k_+ - k_-) = 2 = 2\psi''(u_0)\ell = 2(k_+ - C)\ell,$$

which gives the promised (6.14).

By combining equations (6.11), (6.13), (6.12) and (6.14) respectively, we obtain

$$\ln\left(\frac{k_+}{k_-}\right) = \frac{\text{Area}_{\mathbb{H}}(M)}{2\pi\ell} = \frac{C \text{Area}(M)}{2\pi\ell} = \frac{(k_+ + k_-)(k_+ - k_-)}{2k_+k_-}. \quad (6.16)$$

Now let the function  $f$  be given by

$$f : \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto \ln(x) - \frac{x}{2} + \frac{1}{2x}.$$

A consideration of the derivative of  $f$  suffices to conclude that  $f(x) = 0$  if and only if  $x = 1$  (for  $x \in \mathbb{R}_0^+$ ). It has been shown in (6.16) that  $f(\frac{k_+}{k_-}) = 0$ , whence the conclusion  $k_+ = k_-$  can be drawn. But this contradicts (6.7), and the theorem is proved. ■

**Remark 18.** The spheres are actually a maximum of the constrained functional, as follows from

$$4\pi \text{Area} = \int K \, d\Omega \int d\Omega \geq \left( \int \sqrt{K} \, d\Omega \right)^2 = \left( \text{Area}_{\mathbb{H}} \right)^2.$$

**Remark 19.** We can also state the following corollary of Theorem 17:

**Corollary 20.** *The spheres are the only ovaloids in  $\mathbb{E}^3$  that satisfy  $H_{\mathbb{H}}\sqrt{K} = CH$  for some  $C \in \mathbb{R}$ .*

### 6.3. Third variational problem

The following theorem, which has been mentioned already by M. Wiehe in [30, remark 5.6.(iii)], solves the so-called ‘ $\mathbb{H}$ -isoperimetric problem’, which was formulated in [5; 6] and elsewhere. The proof of the theorem will be presented as well because of its restricted length: it consists of a mere combination of the work of K. Leichtweiss and F. Manhart.

**Theorem 21** (K. Leichtweiss, F. Manhart, M. Wiehe). *If an ovaloid in  $\mathbb{E}^3$  is a*

critical point of the  $\mathbb{I}$ -area with respect to deformations under which the volume is preserved, then it is a sphere.

PROOF OF THEOREM 21. Critical points of this variational problem need to satisfy the relation  $H_{\mathbb{I}}\sqrt{K} = C$  (for some  $C \in \mathbb{R}$ ), or yet

$$\frac{1}{2} \operatorname{trace} A_{(N_{\mathbb{I}})} = C.$$

On the other hand, by an application of [21, Satz 6.1] (see also [28]), it follows that the relative shape operator with respect to Manhart's  $\mathbb{I}$ -normal field is a constant multiple of the identity:

$$\frac{-1}{C} \bar{D}_V N_{\mathbb{I}} = V$$

for all  $V \in \mathfrak{X}(M)$ . Since also  $\bar{D}_V P = V$ , it follows that  $\frac{-1}{C} N_{\mathbb{I}} = P$  with respect to a suitably chosen origin, and in particular,

$$\sqrt{K} = C\rho. \quad (6.17)$$

Since the spheres are the only ovaloids that satisfy the equation (6.17), as has been shown in [14, theorem 1], the proof is finished. ■

**Remark 22.** The spheres are actually a maximum of the constrained functional, as follows from the 'I-isoperimetric inequality' [18]:

$$48\pi^2 \operatorname{Vol} \geq \left( \operatorname{Area}_{\mathbb{I}} \right)^3.$$

**Remark 23.** We can also state the following corollary of Theorem 17:

**Corollary 24.** *The spheres are the only ovaloids in  $\mathbb{E}^3$  that satisfy  $H_{\mathbb{I}}\sqrt{K} = C$  for some  $C \in \mathbb{R}$ .*

**Acknowledgement** It is a pleasure to thank Dr. S. Haesen and Professor U. Simon for useful suggestions. The author was partially supported by the Research Foundation – Flanders (project G.0432.07).

#### REFERENCES

- [1] C. Baikoussis and T. Koufogiorgos, On convex hypersurfaces in Euclidean space, *Archiv der Mathematik* **49** (4) (1987), 337–43.
- [2] D.E. Blair, Spaces of metrics and curvature functionals, In F.J.E. Dillen and L.C.A. Verstraelen (eds), *Handbook of differential geometry*, Vol. I, North-Holland, Amsterdam 2000.
- [3] P.O. Bonnet, Sur une propriété de maximum relative à la sphère, *Nouvelles Annales de Mathématiques* **12** (1853), 433–53.
- [4] E.B. Christoffel, Über die Bestimmung der Gestalt einer krummen Oberfläche durch lokale Messungen auf derselben, *Journal für die reine und angewandte Mathematik (Crelle's Journal)* **64** (1865), 193–209.

- [5] E. Glässner, Über die Minimalflächen der zweiten Fundamentalform, *Monatshefte für Mathematik* **78** (1974), 193–214.
- [6] E. Glässner and U. Simon, Zur Geometrie der zweiten Grundform, *Überblicke Mathematik* **6** (1973), 197–235.
- [7] W.R. Hamilton, On a new method of investigating the relations of surfaces to their normals, with results respecting the curvatures of ellipsoids, *Dublin University Review*, July 1833, 583–4.
- [8] T. Hasanis, A new characterization of the sphere in  $\mathbb{R}^3$ , *Annales Polonici Mathematici* **38** (1980), 47–9.
- [9] C.C. Hsiung, Some integral formulas for closed hypersurfaces, *Mathematica Scandinavica* **2** (1954), 286–94.
- [10] J.-H. Jellett, Sur la surface dont la courbure moyenne est constante, *Journal de Mathématiques Pures et Appliquées*, Série I, **18** (1853), 163–7.
- [11] T. Koufogiorgos and T. Hasanis, A characteristic property of the sphere, *Proceedings of the American Mathematical Society* **67** (1977), 303–305.
- [12] T. Koutroufiotis, Two characteristic properties of the sphere, *Proceedings of the American Mathematical Society* **44** (1974), 176–8.
- [13] W. Kühnel, Conformal transformations between Einstein spaces, In R.S. Kulkarni and U. Pinkall (eds), *Conformal geometry* (Aspects of mathematics: E 12), Vieweg, Wiesbaden, 1988.
- [14] K. Leichtweiss, On a problem of W.J. Firey in connection with the characterization of spheres, *Mathematica Pannonica* **6** (1995) 1, 67–75.
- [15] An-Min Li, Some integral formulas for hypersurfaces and a generalization of the Hilbert-Liebmann theorem, *Proceedings of the American Mathematical Society* **88** (2) (1983), 326–9.
- [16] F. Manhart, Zur Differentialgeometrie der 2. Grundform, Second Austrian geometry colloquium in Rein (Rein, 1983), Ber. No. 219, 7pp.
- [17] F. Manhart, Relativgeometrische Kennzeichnungen euklidischer Hypersphären, *Geometriae Dedicata* **29** (2) (1989), 193–207.
- [18] F. Manhart, Kennzeichnungen Euklidischer Hypersphären durch isoperimetrische Ungleichungen, *Glasnik Matematički. Serija III* **24** (44)(4) (1989), 541–55.
- [19] H. Minkowski, Volumen und Oberfläche, *Mathematische Annalen* **57** (1903), 447–95.
- [20] R. Schneider, Closed convex hypersurfaces with second fundamental form of constant curvature, *Proceedings of the American Mathematical Society* **35** (1972), 230–3.
- [21] U. Simon, Minkowskische Integralformeln und ihre Anwendungen in der Differentialgeometrie im Großen, *Mathematische Annalen* **173** (1967), 307–21.
- [22] U. Simon, Characterizations of the sphere by the curvature of the second fundamental form, *Proceedings of the American Mathematical Society* **55** (1976), 382–4.
- [23] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the affine differential geometry of hypersurfaces*, Lecture Notes, Science University of Tokyo, 1991.
- [24] G. Stamou, Global characterizations of the sphere, *Proceedings of the American Mathematical Society* **68** (1978), 328–30.
- [25] G. Stamou, Characterizations of the sphere by the mean II-curvature, *Bulletin of the Australian Mathematical Society* **23** (2) (1981), 249–53.
- [26] G. Stamou, Characterizations of the Sphere by the II-curvature and the mean II-curvature, *Revue Roumaine de Mathématiques Pures et Appliquées* **32** (1987), 555–9.
- [27] G. Stamou, Kennzeichnungen Euklidischer Hypersphären durch relativgeometrische Größen, *Studia Scientiarum Mathematicarum Hungarica* **40** (3) (2003), 349–58.
- [28] W. Süss, Zur relativen Differentialgeometrie V: über Eihyperflächen im  $\mathfrak{R}^{n+1}$ , *the Tôhoku Mathematical Journal* **30** (1929), 202–209.
- [29] S. Verpoort, The geometry of the second fundamental form: curvature properties and variational aspects, Unpublished PhD dissertation, K.U.Leuven, 2008.  
<http://hdl.handle.net/1979/1779>
- [30] M. Wiehe, *Deformations in affine hypersurface theory*, PhD dissertation, TU Berlin 1998, Shaker Verlag, Aachen, 1999.