ON THE PRODUCT OF DERIVATIONS IN BANACH ALGEBRAS

Nadia Boudi
Département de Mathématiques, Faculté des Sciences,
Université Moulay Ismail, Meknes, Morocco

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Dedicated to Professor El Amin Kaidi on the occasion of his sixtieth birthday

Abstract

The purpose of this paper is to study derivations d, g defined on a complex Banach algebra A such that the spectrum $\sigma(dg(x))$ is finite for all $x \in A$. In particular we show that if the algebra A is semisimple, then $dg(x)^3$ lies in the socle of A for every $x \in A$.

1. Introduction

In [8], Brešar and Šemrl studied commuting pairs d and g of continuous derivations defined on a Banach algebra A such that $dg(x)$ is quasi-nilpotent for all $x \in A$. A short proof of this result can be found in [13]. In [9], Chebotar, Ke and Lee showed that the assumption of commutativity $dg = gd$ is unnecessary. The reader is referred to [8] for the general motivation of this research direction. The present paper continues this line of investigations. Our goal is to show that if d, g are continuous derivations defined on a complex Banach algebra A such that the spectrum $\sigma(dg(x))$ is finite for all $x \in A$, then $(dg(x))^3$ lies in the socle modulo the radical for every $x \in A$. It should be pointed out that our study is closely connected with questions concerning derivations mapping into the socle [5; 6; 7].

2. The case of dense algebras

Let A be an algebra. By $[x, y]$ we denote the commutator $xy - yx$ of $x, y \in A$. For each $x \in A$ let δ_x denote the inner derivation $\delta_x(y) = [x, y]$ implemented by x. If A is semisimple, the socle $\operatorname{soc} A$ of A is defined as the sum of all minimal left ideals of A. If there are no minimal left ideals, then $\operatorname{soc} A = 0$ by definition. The Jacobson radical of A will be denoted by $\operatorname{rad} A$. Let X be a complex vector space

*Corresponding author, e-mail: nadia_boudi@hotmail.com
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and let T be a linear operator on X. The point spectrum of T is $\sigma_p(T) = \{\lambda \in \Im : \lambda I - T$ is not injective $\}$, where as usual, I denotes the identity mapping on X. If F is an arbitrary set, we will write $\sharp F$ for the cardinality of F .

Let U and V be vector spaces over an infinite field $\mathbf{F}$ and let V_0 be a finite dimensional subspace of V. Amitsur [1] proved that if $T_1, \ldots, T_n : U \to V$ are linear operators such that $T_1 u, \ldots, T_n u$ are linearly dependent modulo V_0 for every $u \in U$, then there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that $S = \alpha_1 T_1 + \ldots + \alpha_n T_n$ satisfies

$$\dim SU \leq \dim V_0 + \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - 1.$$

Aupetit [2, p. 86] considered the case that U and V are complex vector spaces and $V_0 = \{0\}$. He proved that S can be chosen so that $\operatorname{rank} S \leq n-1$. Let us point out that the estimate in [2] is sharp. Brešar and Šemrl [8, theorem 2.2] improved Amitsur's result by obtaining a sharp estimate on the rank of S. We refer to [13] for more background on locally linearly dependent operators.

We will make crucial use of the following lemma, which is a consequence of Amitsur's result.

Lemma 2.1. *Let n, r be positive integers, U and V vector spaces over a field F and let V_0 be a finite dimensional subspace of V. Let $T_1, \ldots, T_n : U \to V$ be linear operators. Suppose that for any r- tuple of vectors $\zeta_1, \ldots, \zeta_r$ from U the set $\{T_1\zeta_i, \ldots, T_n\zeta_i : i = 1, \ldots, r\}$ is linearly dependent modulo V_0. Then there exists a nontrivial linear combination of $T_1, \ldots, T_n$ of finite rank.*

Proof. Suppose first that for all $\zeta \in X$ the vectors $T_1\zeta, \ldots, T_n\zeta$ are linearly dependent modulo V_0. Then we are done by [1, lemma 1]. Suppose now that this is not the case and let r be the largest positive integer such that there exist $\zeta_1, \ldots, \zeta_r \in X$ with the property that the set $\{T_1\zeta_i, \ldots, T_n\zeta_i : i = 1, \ldots, r\}$ is linearly independent modulo V_0. Set $V_0' = V_0 + \operatorname{span}\{T_1\zeta_i, \ldots, T_n\zeta_i : i = 1, \ldots, r\}$. Then for every $\zeta \in X$, the vectors $T_1\zeta, \ldots, T_n\zeta$ are linearly dependent modulo V_0'. Applying [1, lemma 1] again, we get the desired conclusion. ■

The main part of the proof of our main result is contained in the following theorem.

Theorem 2.2. *Let X be a complex vector space and $\mathcal{A}$ a dense algebra of linear operators on X. Let T, T' be linear operators on X and let $n \in \mathbb{N}^*$. Suppose that for each $S \in \mathcal{A}$, $\sharp\sigma_p(\delta_T\delta_{T'}(S)) < n$. Then there exist scalars $\lambda, \lambda' \in \mathbb{C}$ such that $\operatorname{rank}(T' - \lambda' I)(T - \lambda I) < \infty$ and one of the following assertions holds:*

(i) *$\operatorname{rank}(T' - \lambda' I)^2 < \infty$ and there exists $\beta \in \mathbb{C}$ such that $\operatorname{rank}(T - \beta I)(T' - \lambda' I) < \infty$,*

(ii) *$\operatorname{rank}(T - \lambda I)(T' - \lambda' I) < \infty$ and either $\operatorname{rank}(T - \lambda I)^2 < \infty$ or $\operatorname{rank}(T' - \lambda' I)^2 < \infty$.*

Moreover, $(\delta_T\delta_{T'}(S))^3$ has finite rank for every $S \in \mathcal{A}$.

PROOF. The proof is long, so it will be broken into several steps. Let us point out that some of our steps are inspired by those of [8] and [9]. Without loss of generality we assume henceforth that X is infinite dimensional. Recall that

$$\delta_T\delta_{T'}(S) = TT'S + ST'T - TST' - T'ST$$

for every $S \in \mathcal{A}$.

Step 1: Assume towards a contradiction that there exist $\zeta_1, \ldots, \zeta_n \in X$ such that the set $\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, ..., n\}$ is linearly independent. Choose $S \in \mathcal{A}$ such that

$$S\zeta_i = ST\zeta_i = ST'\zeta_i = 0 \quad \text{and} \quad ST'T\zeta_i = i\zeta_i \quad (i = 1, ..., n).$$

Then $\delta_T\delta_{T'}(S)\zeta_i = i\zeta_i$, implying that $\sharp\sigma_p(\delta_T\delta_{T'}(S)) \geq n$, a contradiction. Thus, for any n-tuple of vectors $\zeta_1, ..., \zeta_n$ from X, the set $\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, ..., n\}$ is linearly dependent.

Step 2: Suppose that there exist $\zeta_1, \ldots, \zeta_n \in X$ with the property that the set $\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, ..., n-1\} \cup \{\zeta_n, T\zeta_n, T'\zeta_n\}$ is linearly independent. It follows from Step 1 that

$$T'T\zeta_n \in \operatorname{span}(\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, ..., n-1\} \cup \{\zeta_n, T\zeta_n, T'\zeta_n\}).$$

Choose $S \in \mathcal{A}$ such that

$$S\zeta_i = ST\zeta_i = ST'\zeta_i = 0 \quad (i = 1, ..., n)$$

and

$$ST'T\zeta_i = i\zeta_i \quad (i = 1, \ldots, n-1).$$

Then $\delta_T\delta_{T'}(S)\zeta_i = i\zeta_i$, $i = 1, \ldots, n-1$ and $\delta_T\delta_{T'}(S)\zeta_n \in \operatorname{span}\{\zeta_1, \ldots, \zeta_{n-1}\}$. Thus $\{0, 1, \ldots, n-1\} \subseteq \sigma_p(\delta_T\delta_{T'}(S))$, which contradicts our assumption. As a result, for any n-tuple of vectors $\zeta_1, \ldots, \zeta_n$ from X, the set

$$\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, ..., n-1\} \cup \{\zeta_n, T\zeta_n, T'\zeta_n\}$$

is linearly dependent.

Step 3: We claim that if there exists a linear combination of I, T, T' which has finite rank, then the theorem follows. Suppose first that there exists $\lambda \in \mathbb{C}$ such that either $T - \lambda I$ has finite rank or $T' - \lambda I$ has finite rank. Then we are done. Suppose next that for every scalar λ, the operator $T - \lambda I$ does not have finite rank. Replacing T' by $T' - \lambda' I$, for $\lambda' \in \mathbb{C}$ suitable, we can assume that $T' = \alpha T + F$ for some finite rank operator F and nonzero scalar α. Suppose that there exist $\zeta_1, \ldots, \zeta_n \in X$ such that the set $\{\zeta_i, T\zeta_i, T^2\zeta_i : i = 1, \ldots, n\}$ is linearly independent

modulo FX. Take $S \in \mathcal{A}$ such that

$$S\zeta_i = ST\zeta_i = 0, ST^2\zeta_i = i\zeta_i \quad (1 \le i \le n), \text{ and } S \mid FX = 0.$$

Then $\delta_T\delta_{T'}(S)\zeta_i = i\alpha\zeta_i$, $1 \le i \le n$. This entails that $\{\alpha, \ldots, n\alpha\} \subseteq \sigma_p(\delta_T\delta_{T'}(S))$, a contradiction. Consequently, for any n-tuple of vectors $\zeta_1, \ldots, \zeta_n$ from X, the set $\{\zeta_i, T\zeta_i, T^2\zeta_i : 1 \le i \le n\}$ is linearly dependent modulo FX. It follows from Lemma 2.1 that there exists a nontrivial linear combination of I, T, T^2 which has finite rank. Write $T^2 = \mu T + \beta I + F'$, where F' is a finite rank operator and $\beta, \mu \in \mathbb{C}$. Then $(T - \frac{1}{2}\mu I)^2 = (\beta + \frac{1}{4}\mu^2)I + F'$. Replacing T by $T - \frac{1}{2}\mu I$, we can assume that $T^2 = \mu' I + F'$, where F' is a finite rank operator and μ' is a scalar. We have two cases to discuss.

(1) $\mu' \neq 0$. Let J' be a basis of the vector space $FX + F'X + TF'X + TFX$ and suppose that there exist $\zeta_1, \ldots, \zeta_n \in X$ such that the set $\{\zeta_i, T\zeta_i : i = 1, \ldots, n\} \cup J'$ is linearly independent. Then there exists $S \in \mathcal{A}$, such that

$$S\zeta_i = i\zeta_i, ST\zeta_i = 0 \quad (1 \le i \le n), \text{ and } S(J') = \{0\}.$$

This entails that $\delta_T\delta_{T'}(S)T\zeta_i = -2i\alpha\mu' T\zeta_i - i\mu' F\zeta_i$ and $(\delta_T\delta_{T'}(S))F\zeta_i = 0$. Observe that span $\{T\zeta_1, \ldots, T\zeta_n, F\zeta_1, \ldots, F\zeta_n\}$ is invariant under S. By considering the corresponding matrix representation of $\delta_T\delta_{T'}(S)$ with respect to

$$\{T\zeta_1, \ldots, T\zeta_n, F\zeta_1, \ldots, F\zeta_n\},$$

we infer that $-2\alpha\mu', \ldots, -2n\alpha\mu'$ are eigenvalues of $\delta_T\delta_{T'}(S)$, a contradiction. It follows that for any n-tuple of vectors $\zeta_1, \ldots, \zeta_n$ from X, the set $\{\zeta_i, T\zeta_i : i = 1, \ldots, n\} \cup J'$ is linearly dependent. Now Lemma 2.1 yields that there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has finite rank. Clearly, this case cannot occur.

(2) $\mu' = 0$, that is $T^2 = F'$ where F' is a finite rank operator. Then, the operators $TT', T'T$ have finite rank and we are done.

Henceforth we assume that any nontrivial linear combination of I, T and T' has infinite rank.

Step 4: In this step, we will show that there exists a nontrivial linear combination of I, T, T' and $T'T$ which has finite rank. Suppose first that the set $\Gamma = \{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, \ldots, n-1\}$ is linearly independent for some $(n-1)$-tuple of vectors $\zeta_1, \ldots, \zeta_{n-1} \in X$. Then by Step 2, for every $\zeta_n \in X$, the vectors $\zeta_n, T\zeta_n, T'\zeta_n$ are linearly dependent modulo span (Γ). Whence it follows from [1, lemma 1] that a nontrivial linear combination of I, T, T' has finite rank, a contradiction. Thus for every $(n-1)$-tuple of vectors $\zeta_1, \ldots, \zeta_{n-1}$, the set $\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i : i = 1, \ldots, n-1\}$ is linearly dependent. The desired result follows from Lemma 2.1.

Step 5: We claim that there are $\lambda, \lambda' \in \mathbb{C}$ such that the operator $(T'-\lambda' I)(T-\lambda I)$ has finite rank. Indeed, it follows from Step 4 that there exist scalars α, β, δ, not all zero such that the operator

$$\alpha I + \beta T + \delta T' + T'T = (T' + \beta I)(T + \delta I) + (\alpha - \beta\delta)I$$

has finite rank. Replacing T by $T + \delta I$ and T' by $T' + \beta I$, we can assume that there exists $\mu \in \mathbb{C}$ such that $T'T + \mu I = F$, where F is a finite rank operator. Next suppose that $\mu \neq 0$. Let J be a basis of the vector space FX. Suppose that there exist vectors $\zeta_1, \ldots, \zeta_r \in X$ such that the set $\{\zeta_i, T\zeta_i, T'\zeta_i : i = 1, \ldots, r\}$ is linearly independent modulo FX. Then there is $S \in \mathcal{A}$ such that

$$S\zeta_i = i\zeta_i, ST\zeta_i = 0, ST'\zeta_i = iT'\zeta_i \quad (1 \leq i \leq r) \quad \text{and } S \mid FX = 0.$$

This implies that $(\delta_T\delta_{T'}(S))\zeta_i = -i\mu\zeta_i$, for all i. Thus, $\{-\mu, \ldots, -r\mu\} \subseteq \sigma_p(\delta_T\delta_{T'}(S))$. By choosing a suitable r, we get a contradiction. Consequently, there exists a nonzero positive integer r such that the set $\{\zeta_i, T\zeta_i, T'\zeta_i : i = 1, \ldots, r\}$ is linearly dependent modulo FX for every vectors $\zeta_1, \ldots, \zeta_r$ from X. It follows from Lemma 2.1 that a nontrivial linear combination of I, T, T' has finite rank, a contradiction. Thus $\mu = 0$ and the claim follows.

Without loss of generality, we may assume henceforth that $T'T = F$, where F is a finite rank operator.

Step 6: Suppose that there are vectors $\zeta_1, \ldots, \zeta_r$ of X such that the set

$$\{T'\zeta_i, T'^2\zeta_i, TT'\zeta_i, i = 1, \ldots, r\}$$

is linearly independent modulo FX. Choose $S \in \mathcal{A}$ such that

$$ST'\zeta_i = ST'^2\zeta_i = 0,\ SFX = 0 \text{ and } STT'\zeta_i = i\zeta_i \quad (1 \leq i \leq r).$$

Then $(\delta_T\delta_{T'}(S))T'\zeta_i = -iT'\zeta_i$ for each i. This implies that $-i \in \sigma_p(\delta_T\delta_{T'}(S))$ for all $1 \leq i \leq r$. By choosing a suitable r, we get a contradiction. Using Lemma 2.1, we infer that there is a nontrivial linear combination of T', T'^2 and TT' which has finite rank.

Step 7: Either TT' has finite rank or there exists a nontrivial linear combination of T', T'^2 which has finite rank. Indeed, it follows from Step 6 that there exist scalars α', β', μ not all zero such that $\mu TT' = F" + \alpha'T' + \beta'T'^2$, where $F"$ is a finite rank operator. Suppose first that $\mu = 0$, then $\alpha'T' + \beta'T'^2 = -F"$, $(\alpha', \beta') \neq (0, 0)$ and we are done. Suppose now that $\mu \neq 0$. Then there exists $\alpha, \beta \in \mathbb{C}$ such that $TT' = F' + \alpha T' + \beta T'^2$, where F' is a finite rank operator. We have two cases to discuss.

(1) $\alpha \neq 0$. Then

$$(\delta_T\delta_{T'}(S)) = F'S + \alpha T'S + \beta T'^2S + SF - TST' - T'ST.$$

Let J be a basis of the vector space $V_0 = FX + F'X + T'F'X + TF'X$.

Suppose that the set $\{T'\zeta_i, T'^2\zeta_i : 1 \leq i \leq r\} \cup J$ is linearly independent for some vectors $\zeta_1, \ldots, \zeta_r$ from X. Fix $\nu_1, \ldots, \nu_r \in \mathbb{C}$, $\nu_i \neq \nu_j$ if $i \neq j$. Then, there exists $S \in \mathcal{A}$ such that

$$SJ = 0, ST'\zeta_i = 2\nu_i\zeta_i, ST'^2\zeta_i = \nu_i T'\zeta_i \quad (1 \leq i \leq r).$$

This entails that

$$(\delta_T\delta_{T'}(S))T'\zeta_i = -\alpha\nu_i T'\zeta_i + \nu_i F'\zeta_i \quad (1 \leq i \leq r).$$

Observe that $(\delta_T\delta_{T'}(S))F'\zeta_i = 0$ for all i . Set

$$H = \text{span}\,\{F'\zeta_1, \ldots, F'\zeta_r, T'\zeta_1, \ldots, T'\zeta_r\}.$$

Then H is invariant under $\delta_T\delta_{T'}(S)$. By considering the corresponding matrix representation of $\delta_T\delta_{T'}(S)$ with respect to the basis

$$\{T'\zeta_1, \ldots, T'\zeta_r, F'\zeta_1, \ldots, F'\zeta_r\},$$

we infer that $-\alpha\nu_1, \ldots, -\alpha\nu_r$ are eigenvalues of $\delta_T\delta_{T'}(S)$. If $r \geq n$, we get a contradiction. Consequently, for any n-tuple of vectors $\zeta_1, \ldots, \zeta_n$ from X, the set $\{T'\zeta_i, T'^2\zeta_i : i = 1, \ldots, n\}$ is linearly dependent modulo V_0. Lemma 2.1 implies that there exists a nontrivial linear combination of T', T'^2 which has finite rank.

(2) $\alpha = 0$. Then $TT' = F' + \beta T'^2$. Suppose that TT' is not a finite rank operator. Then $\beta \neq 0$. Since $T'T$ is a finite rank operator, T'^3 has finite rank too. Let J be a basis of the vector space $T'^3X + FX + F'X + TF'X + T'F'X$. Suppose that there exist $\zeta_1, \ldots, \zeta_r \in X$ such that the vectors $\{T'^2\zeta_i : i = 1, \ldots, r\} \cup J$ are linearly independent. Then we can choose $S \in \mathcal{A}$ such that $S(J) = \{0\}$ and $ST'^2\zeta_i = i\zeta_i$, $(1 \leq i \leq r)$. This implies that

$$\delta_T\delta_{T'}(S))T'^2\zeta_i = iF'\zeta_i + i\beta T'^2\zeta_i, \;\; \delta_T\delta_{T'}(S))F'\zeta_i = 0, \; (1 \leq i \leq r).$$

As above, we obtain a contradiction. Now it is clear that T'^2 has finite rank. Thus TT' has finite rank too, a contradiction. We have thereby shown that if $\alpha = 0$, then $\beta = 0$ and TT' has finite rank. This completes the proof of this step.

Step 8: Suppose that TT' does not have finite rank. Then we will show next that T'^2 is a finite rank operator and there exists $\beta \in \mathbb{C}$ such that $TT' = \beta T' + F_0$, where F_0 is a finite rank operator. First observe that by Step 7, there is a nontrivial linear combination of T', T'^2 which has finite rank. Write $T'^2 = \lambda T' + F'$, where F' is a finite rank operator.

Let J be a basis of the vector space $FX + F'X + TF'X + T^2F'X$. Suppose that there exist vectors $\zeta_1, \ldots, \zeta_n \in X$ such that the set

$$\{T'\zeta_1, \ldots, T'\zeta_n, TT'\zeta_1, \ldots, TT'\zeta_n\} \cup J$$

is linearly independent. Choose $S \in \mathcal{A}$ such that

$$S(J) = \{0\},\ \ ST'\zeta_i = T'\zeta_i,\ \ STT'\zeta_i = i\zeta_i,\ \ i = 1, \ldots, n.$$

Then $(\delta_T\delta_{T'}(S))T'\zeta_i = -iT'\zeta_i + TF'\zeta_i$, and $(\delta_T\delta_{T'}(S))TF'\zeta_i = 0$, for each i. As above, we obtain a contradiction. Thus, there exists $\gamma, \eta \in \mathbb{C}$ such that $(\gamma, \eta) \neq (0, 0)$ and $\gamma T' + \eta TT'$ has finite rank. Since TT' is not a finite rank operator, we must have $\gamma \neq 0$. Further, it is clear that $\eta \neq 0$. Multiplying by T', we infer that T'^2 is a finite rank operator.

Step 9: Suppose that $\operatorname{rank} TT' < \infty$. We will show that either $\operatorname{rank} T'^2 < \infty$ or $\operatorname{rank} T^2 < \infty$. First, assume that there are vectors $\zeta_1, \ldots, \zeta_n \in X$ such that the set $\{T^2\zeta_1, \ldots, T^2\zeta_n, T'^2\zeta_1, \ldots, T'^2\zeta_n\} \cup J$ is linearly independent, where J is a basis of the vector space $FX + TT'X + T^2T'X$. Choose $S \in \mathcal{A}$ such that $S(J) = \{0\}$ and $ST^2\zeta_i = ST'^2\zeta_i = i\zeta_i,\ 1 \leq i \leq n$. Then there exist $u_1, \ldots, u_n \in TT'X$ such that

$$\delta_T\delta_{T'}(S))(T\zeta_i + T'\zeta_i) = u_i - i(T\zeta_i + T'\zeta_i).$$

Observe moreover that $(\delta_T\delta_{T'}(S))TT'X = 0$. It is now easy to see that $\{-1, \ldots, -n\} \subseteq \sigma_p(\delta_T\delta_{T'}(S))$, a contradiction. Using Lemma 2.1, we infer that a non trivial linear combination of T^2, T'^2 has finite rank. Suppose next that T'^2 does not have finite rank and let $\lambda \in \mathbb{C}$ such that $T^2 + \lambda T'^2 = F'$, where F' is a finite rank operator. If $\lambda = 0$, we are done. Assume now that $\lambda \neq 0$ and suppose that there are vectors $\zeta_1, \ldots, \zeta_n \in X$ such that the set $\{T^2\zeta_1, \ldots, T^2\zeta_n\}$ is linearly independent modulo the vector space $V_0 = FX + TT'X + T^2T'X$. With no loss of generality, we may suppose that $\lambda = -1$. Choose $S \in \mathcal{A}$ such that $S \mid V_0 = 0$ and $ST^2\zeta_i = i\zeta_i$. Then $ST'^2\zeta_i = i\zeta_i$ for each $i \in \{1, \ldots, n\}$ and there exist $u_1, \ldots, u_n \in TT'X$ such that

$$\delta_T\delta_{T'}(S)(T\zeta_i + T'\zeta_i) = u_i - i(T\zeta_i + T'\zeta_i) \quad (1 \leq i \leq n).$$

Since $(\delta_T\delta_{T'}(S))TT'X = 0$, we conclude as above that $\{-1, \ldots, -n\} \subseteq \sigma_p(\delta_T\delta_{T'}(S))$, a contradiction. Thus T^2 has finite rank. This implies that T'^2 has finite rank too. Clearly, this case cannot occur. This completes the proof of this step.

The proof of the theorem is now complete.

■

Remarks. It is easy to construct an example of a complex Banach space X and bounded operators T, T' on X such that $T^2 = TT' = T'T = 0$, but $T'^2 \neq 0$. In particular, this implies that there exists $S \in \mathcal{B}(X)$ with the property that $(\delta_T\delta_{T'}(S))^2 \neq 0$.

Note that using the assumptions and notations as in Theorem 2.2, it follows that $(\delta_T\delta_{T'})^2(S)$ is a finite rank operator for every $S \in \mathcal{A}$.

Let $\mathcal{A}$ be a dense algebra of linear operators on a vector space X. Following the terminology of [8], we shall say that a derivation d of $\mathcal{A}$ is inner if there is a linear operator T on X such that $d(S) = [T, S]$ for every $S \in \mathcal{A}$. Derivations that are not inner are called outer.

Proposition 2.3. *Let $\mathcal{A}$ be a dense algebra of linear operators on a complex vector space X, and let d, g be nonzero derivations of $\mathcal{A}$. Suppose that there exists a positive integer $n \in \mathbb{N}^*$ such that $\sharp\sigma_p(dg(S)) < n$ for all $S \in \mathcal{A}$. Then there are operators T, T' on X such that $d(S) = \delta_T(S)$ and $g(S) = \delta_{T'}(S)$ for all $S \in \mathcal{A}$.*

Proof. We proceed by considering three cases.

Case 1: Suppose that d and g are outer. Assume first that any nontrivial linear combination of d and g is outer. Let $\zeta_1, \ldots, \zeta_n$ be linearly independent vectors in X. Using [3, theorem 1.2] we see that there exists $S' \in \mathcal{A}$ such that

$$S'\zeta_i = 0,\ (d(S'))\zeta_i = i\zeta_i,\ g(S')\zeta_i = 0, \quad (1 \leq i \leq n).$$

Next choose $S \in \mathcal{A}$ such that

$$S(dg(S')\zeta_i) = 0,\ g(S)\zeta_i = \zeta_i, \quad (1 \leq i \leq n).$$

Then

$$(dg(SS'))\zeta_i = [(dg(S))S' + g(S)d(S') + d(S)g(S') + S(dg(S'))]\zeta_i = i\zeta_i,$$

and hence $\{1, \ldots, n\} \subseteq \sigma_p((dg)(SS'))$, a contradiction. Next suppose that there exists a nonzero scalar $\alpha \in \mathbb{C}$ and a linear operator T on X such that $(d+\alpha g)(S) = [T, S]$ for all $S \in \mathcal{A}$. Let $\zeta_1, \ldots, \zeta_n$ be linearly independent vectors in X. It follows from [8, theorem 3.6] that there exists $S' \in \mathcal{A}$ such that

$$S'\zeta_i = S'T\zeta_i = 0,\ (g(S'))\zeta_i = i\zeta_i, \quad (1 \leq i \leq n).$$

Now choose $S \in \mathcal{A}$ such that

$$S(dg(S')\zeta_i) = S\zeta_i = ST\zeta_i = 0,\ g(S)\zeta_i = \zeta_i\ (1 \leq i \leq n).$$

Then $(dg(SS'))\zeta_i = -2\alpha i\zeta_i$ for $1 \leq i \leq n$. This implies that

$$\{-2\alpha, \ldots, -2n\alpha\} \subseteq \sigma_p(dg(SS')),$$

a contradiction. We have thereby shown that this case cannot occur.

Case 2: Suppose that g is outer and there exists a linear operator T on X such that $d(S) = [T, S]$ for all $S \in \mathcal{A}$. We claim that there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has finite rank. Suppose that there exist $\zeta_1, ..., \zeta_n \in X$ such that the set $\{\zeta_i, T\zeta_i : i = 1, \ldots, n\}$ is linearly independent. Applying once again [8, theorem 3.6], we get the existence of an element $S' \in \mathcal{A}$ such that

$$S'\zeta_i = 0,\ S'T\zeta_i = i\zeta_i,\ g(S')\zeta_i = 0.$$

Moreover, there exists $S \in \mathcal{A}$ such that

$$S(dg(S'))\zeta_i = 0,\ g(S)\zeta_i = \zeta_i, \quad (1 \leq i \leq n).$$

Then $(dg(SS'))\zeta_i = -i\zeta_i$. This entails that $\{-1, \ldots, -n\} \subseteq \sigma_p(dg(SS'))$, a contradiction. Thus for any n-tuple of vectors $\zeta_1, \ldots, \zeta_n$ from X , the set $\{\zeta_i, T\zeta_i : i = 1, \ldots, n\}$ is linearly dependent. Now the claim follows from [7, lemma 3.1]. Observe that every element of $d(\mathcal{A})$ is a finite rank operator, consequently, $\mathcal{A}$ contains finite rank operators. From [12, p. 87], we infer that there exists a linear operator T' on X such that $g = \delta_{T'}$, a contradiction.

Case 3: d is outer and there exists $T' \in \mathcal{L}(X)$ such that $g(S) = \delta_{T'}(S)$ for all $S \in \mathcal{A}$. Suppose that the set $\{\zeta_i, T'\zeta_i : i = 1, \ldots, n\}$ is linearly independent for some vectors $\zeta_1, \ldots, \zeta_n \in X$. Then it follows from [8, theorem 3.6] that there exists $S' \in \mathcal{A}$ such that

$$S'\zeta_i = S'T'\zeta_i = 0,\ (dS')\zeta_i = i\zeta_i,\ (1 \leq i \leq n).$$

Moreover, we can find $S \in \mathcal{A}$ such that $S\zeta_i = S(dg(S'))\zeta_i = 0,\ ST'\zeta_i = \zeta_i,\ (1 \leq i \leq n)$. We have $(dg(SS'))\zeta_i = -i\zeta_i$. As above, we get a contradiction.

Thus, d and g are inner and the proof is complete. ■

The operators T and T' produced in the above proposition do not belong to $\mathcal{A}$. However, we have the following results.

Lemma 2.4. *Let X be a complex vector space and $\mathcal{A}$ a dense algebra of linear operators on X. Let T be a linear operator on X such that $\delta_T(\mathcal{A}) \subset \mathcal{A}$. Suppose that the operator T^2 has finite rank, then $T^2 \in soc\mathcal{A}$.*

PROOF. In the case that X is finite dimensional, there is nothing to prove. So suppose that X is infinite dimensional. Suppose first that there exists $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ has finite rank. Then $\lambda = 0$ since X is infinite dimensional and T^2 has finite rank. By [4, lemma 1.2], T lies in the socle of $\mathcal{A}$ and we are done. Suppose now that for every scalar $\lambda, T - \lambda I$ is not a finite rank operator. Write $T^2 = \sum_{i=1}^{r} u_i \otimes f_i$ for a linearly independent set $\{u_1, \ldots, u_r\}$ of vectors in X and linear functionals $f_1, \ldots, f_r$ on X. It follows from [7, lemma 3.1] that for every $n \in \mathbb{N}^*$, there exist $v_1, \ldots, v_n \in X$ such that the set $\{v_1, \ldots, v_n, Tv_1, \ldots, Tv_n\}$ is linearly independent modulo span $\{u_1, \ldots, u_r, Tu_1, \ldots, Tu_r\}$. Choose vectors $v_1, \ldots, v_r \in X$ such that $\{v_1, \ldots, v_r, Tv_1, \ldots, Tv_r\}$ is linearly independent modulo

$$\text{span}\,\{u_1, \ldots, u_r, Tu_1, \ldots, Tu_r\}.$$

It is easy to see that there exists $S \in \mathcal{A}$ such that $Su_i = v_i$, and $STu_i \in$ span $\{v_1, \ldots, v_r\}$. Further, choose $S' \in \mathcal{A}$ such that $S'u_i = S'Tu_i = S'v_i = 0$ and $S'Tv_i = u_i$. Recall that $(\delta_T)^3(S) = T^3S - 3T^2ST + 3TST^2 - ST^3$, for all $S \in \mathcal{A}$. Since

$$(\delta_T)^3(S)\zeta = \sum f_i(S\zeta)Tu_i - 3\sum f_i(ST\zeta)u_i + 3\sum f_i(\zeta)TSu_i - \sum f_i(\zeta)STu_i,$$

for all $\zeta \in X$, we have $\frac{1}{3}S'\delta_{T^3}(S) = T^2 \in \mathcal{A}$. This completes the proof. ■

Lemma 2.5. *Let X be a complex vector space and $\mathcal{A}$ a dense algebra of linear operators on X. Let T, T' be linear operators on X such that $\delta_T(\mathcal{A}) \subset \mathcal{A}$ and $\delta_{T'}(\mathcal{A}) \subset \mathcal{A}$. Suppose that the operators $T'T, T'^2, TT'$ have finite rank, then $T'T, TT' \in soc\mathcal{A}$.*

Proof. Once again, we can suppose for the proof that X is infinite dimensional. It follows from Lemma 2.4 that $T'^2 \in \mathcal{A}$. Since $\delta_T\delta_{T'} - \delta_{T'}\delta_T = \delta_{TT'-T'T}$, we have $\delta_{TT'-T'T}(\mathcal{A}) \subseteq \mathcal{A}$ and [4, lemma 1.2] yields that $TT' - T'T \in \mathcal{A}$. On the other hand, for each $S \in \mathcal{A}$ we have

$$(\delta_{T'})^2\delta_T(S) = T'^2(TS - ST) - 2T'TST' + 2T'STT' + (TS - ST)T'^2.$$

Thus $T'STT' - T'TST' \in \mathcal{A}$ for all $S \in \mathcal{A}$. Set $T'T = F + TT'$ for some finite rank operator F on X, then $T'STT' - TT'ST' - FST' \in \mathcal{A}$ for all $S \in \mathcal{A}$. Write $TT' = \sum_{i=1}^{r} u_i \otimes f_i$ for a linearly independent set $\{u_1, \ldots, u_r\}$ of vectors in X and linear functionals $f_1, \ldots, f_r$ on X. Then for each $\zeta \in X$ we have

$$(T'STT' - TT'ST' - FST')\zeta = \sum f_i(\zeta)T'Su_i - \sum f_i(ST'\zeta)u_i - FST'\zeta.$$

Let $V_0 = \operatorname{span}\{u_1, \ldots, u_r\} + FX$. Suppose first that there exists $\lambda \in \mathbb{C}$ such that $T' - \lambda I$ has finite rank. Then it follows from [4, lemma 1.2] that $T' - \lambda I \in \mathcal{A}$. But $T'^2 \in \mathcal{A}$ and X is infinite dimensional, thus $\lambda = 0$ and $T' \in \operatorname{soc}(\mathcal{A})$. Now it is obvious that $TT' \in \operatorname{soc}(\mathcal{A})$. Since $F \in \mathcal{A}$, we also have $T'T \in \operatorname{soc}(\mathcal{A})$. Suppose next that for every scalar λ, $T' - \lambda I$ is not a finite rank operator. It follows from Lemma 2.1 that there exist $v_1, \ldots, v_r \in X$ such that $\{v_1, \ldots, v_r, T'v_1, \ldots, T'v_r\}$ is linearly independent modulo V_0. Choose $S, S' \in \mathcal{A}$ such that

$$Su_i = v_i,\ S'T'v_i = u_i \quad (1 \le i \le r), \quad S' \mid V_0 = 0.$$

Then $S'(T'STT' - TT'ST' - FST') = TT' \in \mathcal{A}$. This completes the proof. ■

3. The main result

We will denote the set of all primitive ideals in A by $\operatorname{Prim}(A)$. Recall that primitive ideals are the kernels of irreducible representations of A. In the case that $\operatorname{Prim}(A)$ is nonempty, we will use the following result [14, theorem 2.2.9]:

$$\sigma(x) \cup \{0\} = \cup_{P \in \operatorname{Prim}\ (A)} \sigma(x + P) \cup \{0\}.$$

Lemma 3.1. *Let A be a complex Banach algebra and d, g be continuous derivations on A. Assume that $\sharp\sigma(dg(x)) < n$ for every $x \in A$. Then there exists finitely many distinct primitive ideals $P_1, \ldots, P_r$ such that for each $1 \le i \le r$, the set $\{x \in A : dg(x) + P_i$ is not quasi-nilpotent$\}$ is not empty.*

Proof. Suppose that there are distinct primitive ideals $P_1, \ldots, P_r$ of A and r elements $x_1, \ldots, x_r \in A$, such that $dg(x_i) + P_i$ is not quasi-nilpotent for each $i = 1, \ldots, r$. Let $\pi_1, \ldots, \pi_r$ be irreducible representations of A on Banach spaces

$X_1, \ldots, X_r$ respectively, such that $\operatorname{Ker} \pi_i = P_i$ for all i. According to [10, p. 271], $d(P_i) \subseteq P_i$ and $g(P_i) \subseteq P_i$ for each i. Let us denote by d_i, g_i the derivations on $\pi_i(A)$ induced respectively by d, g. Thus, $\pi_i d = d_i \pi_i$ and $\pi_i g = g_i \pi_i$ for all i. Fix $i \in \{1, \ldots, n\}$. For every $x \in A$, we have $\sigma(d_i g_i \pi_i(x)) \subseteq \sigma(dg(x))$. Thus, $\sharp\sigma(d_i g_i \pi_i(x)) < n$ for every $x \in A$. It follows from Proposition 2.3 and Theorem 2.2 that $(d_i g_i(\pi_i(x)))^3$ has finite rank for each $x \in A$. Next choose a nonzero scalar $\alpha \in \sigma((d_i g_i(\pi_i(x_i)))^3)$. Then α is an eigenvalue of $(d_i g_i(\pi_i(x_i)))^3$. We check easily that there exists $\mu \in \Im$ such that $\mu^3 = \alpha$ and μ is an eigenvalue of $d_i g_i(\pi_i(x_i))$. We have thereby shown that $d_i g_i(\pi_i(x_i))$ has a nonzero eigenvalue. Replacing x_i by λx_i, for $\lambda \in \Im$ suitable, we can assume that $i \in \sigma_p(d_i g_i(\pi_i(x_i)))$. Pick $\zeta_i \in X_i$ such that $d_i g_i(\pi_i(x_i))\zeta_i = i\zeta_i$. According to Proposition 2.3 there exist linear operators T_i, T_i' on X such that $d_i \pi_i(a) = [T_i, \pi_i(a)]$ and $g_i \pi_i(a) = [T_i', \pi_i(a)]$ for every $a \in A$. Set $V_i = \operatorname{span}\{\zeta_i, T\zeta_i, T'\zeta_i, T'T\zeta_i\}$. By the extended Jacobson density theorem [11, p. 283], there exists $a \in \mathcal{A}$ such that $\pi_i(a)u_i = \pi_i(x_i)u_i$ for every $u_i \in V_i$ and for each $i \in \{1, \ldots, r\}$. This entails that $d_i g_i(\pi_i(a))\zeta_i = (d_i g_i(\pi_i(x_i))\zeta_i = i\zeta_i$. Wherefore $\{1, \ldots, r\} \subset \cup_i \sigma(d_i g_i \pi_i(a)) \subseteq \sigma(dg(a))$. As a result, we have $r < n$. This completes the proof. ■

We are now in position to state the main result of this paper.

Theorem 3.2. *Let A be a complex Banach algebra and let d, g be continuous derivations on A. Assume that $\sharp\sigma(dg(x)) < \infty$ for every $x \in A$. Then $(dg(x))^3 + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ for every $x \in A$.*

PROOF. According to [7, lemma 2.1], there exists $n \in \mathbb{N}^*$ such that $\sharp\sigma(dg(x)) < n$ for every $x \in A$. Fix $x \in A$ and let P be a primitive ideal of A. Let π be an irreducible representation of A on a Banach space X such that $\operatorname{Ker} \pi = P$. As in the proof of the above lemma, let us consider d_P, g_P the derivations on $\pi(A)$ induced respectively by d, g. We have as above $\operatorname{rank}(d_P g_P(\pi(x)))^3 < \infty$ so that $(dg(x))^3 + P$ lies in the socle of A/P. Furthermore, it follows from the above lemma that $dg(x) + P$ is quasinilpotent for all but finitely many primitive ideals $P_1, \ldots, P_r$ in $\operatorname{Prim}(A)$. Using [9], we infer that $dg(x)^3 \in P$ for every $P \in \operatorname{Prim}(A)$ such that $P \notin \{P_1, \ldots, P_r\}$. Now, the desired conclusion follows from [7, proposition 2.2]. ■

Remark. It should be noted that using the assumptions and notations as in Theorem 3.2, we can deduce similarly that $(dg)^2(x) + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ for every $x \in A$.

As an immediate consequence of our results we have the following statement.

Corollary 3.3. *Let A be a complex Banach algebra and let d be a continuous derivation on A. Suppose that $\sharp\sigma(d^2(x)) < \infty$ for every $x \in A$. Then $(d^2(x))^2 + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ for every $x \in A$.*

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