

FORCING A FINITE GROUP TO BE ABELIAN

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This paper is dedicated to the memory of Nicolae Popescu (1937–2010)

ABSTRACT

Conditions ensuring that a finite group G is abelian are given in terms of element centralisers and automorphisms of G .

1. Introduction

Let G denote a finite group. There are many ways of forcing G to be abelian. For example, if the maps $x \rightarrow x^k$ with fixed $k \in \{-1, 2, 3\}$ are automorphisms of G , then G is abelian.

A well-known result of Burnside states that if G admits an automorphism τ of order 2 that fixes only the identity element, then G is abelian of odd order. This was extended recently by the first and last named authors, who proved ([3]) that a finite group having a fixed-point-free automorphism in the Fitting subgroup of its automorphism group must be abelian.

H. Zassenhaus [15] discovered an elegant theorem: G is abelian if and only if $N_G(A) = C_G(A)$ for all abelian subgroups A of G . Since it seems that in practice no one would check that G is abelian by using Zassenhaus' theorem, one may ask what

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is the point in proving results like this? The answer is that they have theoretical significance: Zassenhaus used his theorem to show that finite division rings are fields and in fact, later on, we will use Zassenhaus' theorem to find a characterisation of those finite groups in which every subgroup is the fixed point subgroup of a coprime automorphism. But let's pause for a moment in order to analyse the condition in Zassenhaus' theorem.

The first remark is that Zassenhaus' theorem fails to remain valid for infinite groups: the so-called Tarski monsters (these are infinite simple groups of exponent p , where p is a large prime) are counterexamples. It then appears the natural question of replacing "abelian" by something else. If one tries to replace "abelian" with "cyclic", one quickly gets dissatisfied, for $G = A_4$ is a nonabelian group satisfying the condition $N_G(\langle x \rangle) = C_G(\langle x \rangle)$ for every $x \in G$.

Another family of subgroups of G is $\mathcal{S} = \{C_G(x) \mid x \in G\}$. It is easy to check that $|\mathcal{S}| \leq |\{A \mid A \leq G, A' = 1\}|$ and one may ask whether or not the condition $N_G(C_G(x)) = C_G(C_G(x))$ for all $x \in G$ does imply that G is abelian. It does. But it does it in a dissatisfyingly trivial way: for if one takes $x \in Z(G)$, then the condition gives $G = Z(G)$. So the real question here is whether or not $N_G(C_G(x)) = C_G(C_G(x))$ for all $x \in G \setminus Z(G)$ is possible at all. The answer is given in

Theorem 1.1. *If G is a finite nonabelian group, then there exists some element $x \in G \setminus Z(G)$ such that $C_G(C_G(x)) < N_G(C_G(x))$.*

As a direct consequence we have the following Zassenhaus type result:

Corollary 1.1. *If G is a finite group, then G is abelian if and only if $N_G(C_G(x)) = C_G(C_G(x))$ for all elements $x \in G \setminus Z(G)$.*

We now consider another way of forcing G to be abelian, but before starting we need a word about notation. The notation is largely standard and follows that of [8]. When dealing with automorphisms, we will use the exponential notation as a handy functional notation. That is, if $x \in G$ and if $\alpha \in \text{Aut}(G)$ we will denote the image $\alpha(x)$ by x^α .

Let $I = \text{Inn}(G)$ denote the group of inner automorphisms of G and let $J = J(G)$ denote the group of the so-called class-preserving automorphisms of G . Here $\tau \in J$ provided that for every $x \in G$ there exists some $g \in G$ (depending on x) such that $x^\tau = x^g$. It is now clear that $I \leq J \trianglelefteq A = \text{Aut}(G)$, but is not trivial to construct examples of finite groups satisfying $I < J$. The first such example is due to Burnside [1] and later G.E. Wall [14] constructed infinitely many examples. More on $J(G)$ can be found in M. Hertweck's paper [9].

Note that G is abelian if and only if $I = J = 1$. But there are nonabelian p -groups satisfying $I < J = A$ – for a recent set of examples see Malinowska [12], so J could be as large as possible, i.e. it could be equal to the full automorphism group of G .

It is well-known that there exist groups G satisfying $I \leq \Phi(A)$ – here $\Phi(G)$ denotes the Frattini subgroup of G . In fact, the combined results of Gaschütz [6]

and Eick [5] show that if G is a finite group, then there exists some group X satisfying $G \triangleleft X$ and $G \leq \Phi(X)$ if and only if $I \leq \Phi(A)$.

The above remarks suggest the natural problem of determining those finite groups verifying $I \leq \Phi(J)$. The answer is right on our alley:

Theorem 1.2. *Let G be a finite group. Then G is abelian if and only if $\text{Inn}(G) \leq \Phi(J(G))$.*

The proof of Theorem 1.2 is short and illustrates nicely the power of the so-called generalised Frattini argument. But the proof uses properties of the Frattini subgroup that fail to hold for infinite groups, so it seems that Theorem 1.2 is not valid for infinite groups. In order to get a counterexample, it would suffice to construct an infinite complete group that has no maximal subgroups, but this is beyond the scope of this paper. Theorem 1.2 and the cited result of Gaschütz imply that the nonabelian finite groups G satisfying $J(G) = \text{Aut}(G)$ cannot be Frattini subgroups of any finite group.

We now close the circle and we consider an extreme situation that is in some sense the opposite of that in Burnside's theorem cited in the beginning. Call an automorphism τ of G a coprime automorphism if the order of τ is coprime to the order of G . The question is to determine all finite groups satisfying the condition that every abelian subgroup is the fixed point subgroup of some coprime automorphism of G . The cyclic groups of odd prime order are examples of such groups and the (cyclic) group of order 15 is an example as well. The answer is given in

Theorem 1.3. *Let G be a finite group. Then every abelian subgroup of G is the fixed point subgroup of some coprime automorphism of G if and only if G is an abelian group of odd square-free exponent.*

2. The proofs

PROOF OF THEOREM 1.1. In order to simplify notation, write $Z = Z(G)$, $C(x) = C_G(x)$ and $N(C(x)) = N_G(C_G(x))$. The proof goes by contradiction, so assume that G is nonabelian and $N(C(x)) = C(C(x))$ for every $x \in G \setminus Z$. We proceed in a sequence of steps:

Step. 1. $C(x)$ is abelian for every $x \in G \setminus Z$.

Indeed, $C(x) \leq N(C(x)) = C(C(x))$.

Step. 2. If $x, y \in G \setminus Z$, then either $C(x) = C(y)$, or $C(x) \cap C(y) = Z$.

Let $x, y \in G \setminus Z$ such that $C(x) \neq C(y)$ and suppose there exists $w \in C(x) \cap C(y)$, $w \notin Z$. Since $w \in C(x)$ and since by Step 1 $C(x)$ is abelian, $C(x) \leq C(w)$. But $C(w)$ being abelian, $C(w) \leq C(x)$, giving $C(w) = C(x)$. Similarly, $C(w) = C(y)$, so $C(x) = C(y) = C(w)$, a contradiction. This shows that $C(x) \cap C(y) \leq Z$ whenever

$C(x) \neq C(y)$. The inclusion $Z \leq C(x) \cap C(y)$ being obvious, the proof of Step 2 is complete.

Step. 3. If $x \in G \setminus Z$, then $N(C(x)) = C(x)$.

Let $y \in N(C(x)) = C(C(x))$. If $y \in Z$, then $y \in C(x)$, so assume $y \notin Z$. Since $y \in C(C(x))$, one obtains $C(x) \leq C(y)$. And since $C(y)$ is abelian by Step 1, $C(y) \leq C(x)$, giving $C(x) = C(y)$. Thus $y \in C(y) = C(x)$, showing that $N(C(x)) \leq C(x) \leq N(C(x))$ and the desired equality $N(C(x)) = C(x)$ follows.

Step. 4. The contradiction.

Let $x \in G \setminus Z$ and let $U = \bigcup_{g \in G} C(x^g) = \bigcup_{g \in G} C(x)^g$. Since $C(x) < G$, it is well-known (see for example Problem 1.9 of [4]) that $U \subset G$. Pick $y \in G \setminus U$. Then $y \notin Z$ and we have that $V = \bigcup_{g \in G} C(y)^g \subset G$.

By Step 3, $|U \setminus Z| = |G : C(x)|(|C(x)| - |Z|) = |G| - |G : C(x)||Z| \geq |G|/2$, the last inequality following from $|C(x)| \geq 2|Z|$. Similarly, $|V \setminus Z| \geq |G|/2$.

Now note that $U \cap V = Z$, for if $w \in U \cap V$, then $w \in C(x^{g_1}) \cap C(y^{g_2})$ for some $g_1, g_2 \in G$ and $C(x^{g_1}) \cap C(y^{g_2}) = Z$ by step 2. The three sets $U \setminus Z$, $V \setminus Z$ and Z are thus pairwise disjoint. This gives the final contradiction: $|G| \geq |U \setminus Z| + |V \setminus Z| + |Z| \geq |G|/2 + |G|/2 + 1 = |G| + 1$. ■

Remarks.

- (1) Theorem 1.1 fails to be true for infinite groups, the Tarski monsters being again counterexamples. This is due to the fact that when G is infinite and $H < G$ it is possible to have $\bigcup_{g \in G} H^g = G$, so Step 4 of the proof is not valid.
- (2) The condition in Theorem 1.1 seems a bit strong and one may ask what happens if one insists that merely $N_G(C_G(x)) = C_G(x)$ for all $x \in G$. These authors don't know and they leave this as an open problem to the reader. But if this condition implies indeed that G is abelian, then this would provide an extension of a result of I.D. MacDonald [11], see also Aufgabe 15, p. 286 of [10]: If G is a finite nonabelian group, then there exist $x, y \in G$ such that $[x, y] \neq 1$ and $[x, x^y] = 1$.

To see that this is indeed the case, one has to translate MacDonald's theorem into a different language. For $h \in H \leq G$, consider the set $A(h, H, G) = \{g \in G \mid h^g \in H\}$. It is clear that $N_G(H) = \bigcap_{h \in H} A(h, H, G)$. So MacDonald's theorem says that if G is a finite nonabelian group, there exists some $x \in G$ satisfying $C_G(x) \subset A(x, C_G(x), G)$.

Now assume that $N_G(C_G(x)) = C_G(x)$ for all $x \in G$ implies that G is abelian. Then if G is nonabelian, there must exist some x satisfying $C_G(x) < N_G(C_G(x)) \subseteq A(x, C_G(x), G)$, a slight improvement of MacDonald's result.

PROOF OF THEOREM 1.2. Let $I = \text{Inn}(G)$, let J denote the group of class-preserving automorphisms and for an element $x \in G$ let $C_J(x) = \{\alpha \in J \mid x^\alpha = x\}$. It is easy to see that $C_J(x) \leq J$. If G is abelian, then clearly $I = J = \Phi(J) = 1$.

Conversely, assume that $I \leq \Phi(J)$. We want to show that G is abelian. For this, it would suffice to prove that $J = C_J(x)$ for every $x \in G$. For if $J = C_J(x)$ then all elements of J would fix the element x , meaning that $x \in Z(G)$.

Let $Cl_G(x) = \{x^g | g \in G\}$ denote the conjugacy class of x in G . The key remark is that both I, J act transitively on $Cl_G(x)$. So, if $\alpha \in J$, then there exists some $\tau \in I$ such that $x^\alpha = x^\tau$. But then $\alpha\tau^{-1}$ fixes x , thus $\alpha\tau^{-1} \in C_J(x)$. This shows that $\alpha \in \tau C_J(x)$. Since this is true for all $\alpha \in J$, it follows that $J \leq IC_J(x) \leq J$. Thus $J = IC_J(x)$ and since G (hence J) is finite and $I \leq \Phi(J)$ we derive (see for example theorem 5.1.1 of [8]) that $J = C_J(x)$. This completes the proof. ■

PROOF OF THEOREM 1.3. This proof depends essentially on a deeper result on coprime automorphisms. Let G be a finite group and let $\tau \in \text{Aut}(G)$ be a coprime automorphism of G . If $C_G(\tau)$ is the group of fixed points of τ in G and if $[G, \tau] = \langle \{x^{-1}x^\tau | x \in G\} \rangle$, then it is well-known (and easy to prove by using theorem 5.3.5 of [8]) that $G = [G, \tau]C_G(\tau)$. This is the basic tool to complete the proof.

Now assume that $\tau \in \text{Aut}(G)$ is a fixed coprime automorphism of G and let $H \leq C_G(\tau)$. Then $N_G(H)$ is τ -invariant and a short calculation shows that actually $[N_G(H), \tau] \leq C_G(H)$. Applying the above paragraph to the group $N_G(H)$ and to its coprime automorphism $\tau|_{N_G(H)}$ one obtains that $N_G(H) = [N_G(H), \tau]C_{N_G(H)}(\tau) \leq C_G(H)C_{N_G(H)}(\tau) \leq N_G(H)$, giving the equality $N_G(H) = C_G(H)C_{N_G(H)}(\tau)$.

All things we need being now in place, we can start now to prove the theorem. Let B be an arbitrary abelian subgroup of G and suppose that $B = C_G(\tau)$ for some coprime automorphism τ of G . Then by the above paragraph we see that $N_G(B) = C_G(B)C_{N_G(B)}(\tau)$. But $B = C_{N_G(B)}(\tau)$, so we have $N_G(B) = C_G(B)B$. Since B is abelian, $B \leq C_G(B)$, whence $N_G(B) = C_G(B)$. Since B is arbitrary, we are in a position to apply Zassenhaus' theorem cited in the introduction in order to derive that G is in fact abelian.

Let now $1 < H < G$ and let τ be a coprime automorphism of G such that $H = C_G(\tau)$. An easy adaptation of theorem 5.2.3 of [8] shows that $G = [G, \tau] \times H$. Thus every subgroup of the abelian group G is complemented. This could happen only if $\Phi(G) = 1$ and $\Phi(G) = 1$ precisely when G has square-free exponent. So we only have to prove that $|G|$ is odd.

Suppose by contradiction that $|G|$ is even and let H be a subgroup of G of index 2. Write $G = H \times K$, where $K = \langle k \rangle$ and $|k| = 2$. By hypothesis, there exists some coprime automorphism τ of G such that $H = C_G(\tau)$. Then $h^\tau = h$ for all $h \in H$ and $k^\tau = uk$ for some $u \in H$, where u is an element of order at most 2. Now $k^{\tau^2} = u(uk) = u^2k = k$, which shows that τ has order 2. This is a contradiction with $|G|$ even.

Conversely, let G be a finite abelian group of odd square-free exponent and let $H < G$. Write $G = H \times K$ for some subgroup K of G and define an automorphism α of G as follows: α acts trivially on H and by inversion on K . Then this is an automorphism of order 2 of G whose fixed-point-subgroup is H and the proof is complete. ■

Remarks. A harder problem is to determine those finite groups G satisfying the condition that every subgroup is the fixed point subgroup of some automorphism.

Since the trivial subgroup is the fixed point of some automorphism of G , these groups must be solvable by a result of Rowley [13]. Cyclic groups of odd order are examples, see for example [2] and another class of examples is given by those finite groups (described in Gaschütz [7]) in which every subgroup is the centraliser of some element.

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