

THE MAXIMAL EXPECTED LIFETIME OF BROWNIAN MOTION

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ABSTRACT

It is shown that there are domains that maximise the expected lifetime of Brownian motion over all simply connected domains of given inradius and all starting points.

1. Background, and statement of the main result

The expected exit time of Brownian motion from a simply connected planar domain cannot be large unless the domain contains a large disk. This very intuitive property has had many applications, particularly to eigenvalues of the Laplacian. The exit time of Brownian motion from a domain D will be denoted by τ_D and the expectation with respect to Wiener measure for paths with initial point z will be denoted by \mathbf{E}_z . We will write R_D for the supremum radius of all disks contained in D . This is the same as

$$R_D = \sup\{\delta_D(z) : z \in D\},$$

where $\delta_D(z)$ denotes the distance from the point z in D to the boundary of D . This geometric quantity is called the *inradius* of D . It follows from the argument in [6] that there exists a constant C such that

$$\sup_{z \in D} \mathbf{E}_z \tau_D \leq CR_D^2 \tag{1.1}$$

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for all simply connected domains D in the plane. In particular, it is shown in [5] that this inequality holds with $C = 8.62$. In [3] it is shown that

$$\sup_{z \in D} \mathbf{E}_z \tau_D \leq (3.228)R_D^2. \quad (1.2)$$

We thus set

$$\mathcal{A} = \sup \left\{ \frac{\mathbf{E}_z \tau_D}{R_D^2} : D \text{ simply connected, } z \in D \right\} \quad (1.3)$$

so that \mathcal{A} is the best constant C in the inequality (1.1). With the help of an example also constructed in [3], the bounds $1.584 \leq \mathcal{A} \leq 3.228$ hold. An example of Ortega-Cerdà and Pridhnani [17] improves the lower bound for \mathcal{A} to 1.6707.

In 1965, Makai [16] proved that the smallest Dirichlet eigenvalue λ_D for the Laplacian in a simply connected planar domain D satisfies

$$\lambda_D \geq \frac{\mathcal{B}}{R_D^2}. \quad (1.4)$$

The number \mathcal{B} in (1.4) is the best possible, in the sense that it is the largest number for which such an inequality holds for all simply connected domains. Makai's result, which answered an old problem of Polya and Szegő posed in [20, p. 16], was obtained independently by Hayman [12], who showed that $\mathcal{B} > 1/900$. A third proof was found by Osserman [18] showing that $\mathcal{B} \geq 1/4$, which was, incidentally, the numerical bound obtained by Makai over ten years earlier and subsequently lost from view. The addendum to [3] comprises a short note on this history.

The proof in [3] that $\mathcal{A} < 3.228$, together with the inequality

$$\lambda_D \geq \frac{2}{\sup_{z \in D} \mathbf{E}_z \tau_D}$$

led to $\mathcal{B} > 0.619$, which was an improvement on the best numerical bound at the time, namely $\mathcal{B} > 0.25$. To the best of our knowledge this is still the best available lower bound for the eigenvalue in terms of the inradius. A key ingredient in the proof that $\mathcal{A} < 3.228$ was the connection with a third classical inequality of this type, which involved the univalent Bloch-Landau constant \mathcal{U} . This constant was introduced in 1929 by Landau [15], a few years after Bloch's famous paper [8], where he introduced what came to be known as the Bloch constant. (For a more detailed account on how these problems are connected and their relationship to Hardy's inequalities, we refer the reader to [2].)

Suppose that f is a univalent (that is analytic and one-to-one) function defined in the unit disk \mathbb{D} in the complex plane ($\mathbb{D} = \{z: |z| < 1\}$). Then

$$R_{f(\mathbb{D})} \geq \mathcal{U}|f'(0)|. \quad (1.5)$$

In words, the image of the unit disk under a conformal mapping f contains a disk of each radius less than $\mathcal{U}|f'(0)|$. The number \mathcal{U} , known as the univalent or schlicht

Bloch-Landau constant, is the optimal constant in (1.5), in that

$$\mathcal{U} = \inf \left\{ \frac{R_{f(\mathbb{D})}}{|f'(0)|} : f \text{ is univalent in } \mathbb{D} \right\}. \quad (1.6)$$

It follows from the Koebe 1/4-Theorem that $\mathcal{U} \geq 1/4$, while Landau himself proved $\mathcal{U} > 0.566$ in [15]. More recent estimates are $\mathcal{U} \geq 0.57088$ due to Jenkins [13] and also stated by Zhang [22], and $\mathcal{U} > 0.570884$ in Xiong [21].

The infimum in (1.6) is attained. If f is univalent in \mathbb{D} and $R_{f(\mathbb{D})} = \mathcal{U}|f'(0)|$ then the domain $D = f(\mathbb{D})$ is said to be extremal for the univalent Bloch-Landau constant. R.M. Robinson [19] was the first to write down explicitly an argument proving that such extremal domains exist.

In this paper, we adapt Robinson's argument to prove that there are also extremal domains for the inequality (1.1). We have been unable to prove that there are extremal domains for the inequality (1.5) for the first Dirichlet eigenvalue, which does not behave as well as the expected lifetime under conformal mapping. This remains a very interesting problem in its own right. Our main theorem here is:

Theorem 1. *There exists a simply connected domain D of finite inradius R_D and a point $z \in D$ for which*

$$E_z \tau_D = \mathcal{A} R_D^2.$$

In [14], Jenkins described a condition that must be satisfied by any extremal domain for the univalent Bloch-Landau inequality (1.5), that is a domain for which $R_{f(\mathbb{D})} = \mathcal{U}|f'(0)|$. Jenkins [13] also proved that an extremal domain must contain an extremal disk, that is, a disk of radius \mathcal{U} . Jenkins' condition was extended by the second author in [9]. In order to describe this more general condition, we need the notion of harmonic symmetry.

A simple C^1 arc γ is said to be an *internal boundary arc* for a domain D if γ is part of the boundary of D and if, to each non-endpoint ζ of γ , there corresponds a positive ϵ such that the disk with centre ζ and radius ϵ is part of $D \cup \gamma$. Let $z_0 \in D$. Each subarc $\tilde{\gamma}$ of γ has a certain harmonic measure at z_0 , being the probability that Brownian motion starting at z_0 exits D through $\tilde{\gamma}$. This probability can be broken down further if we distinguish between those paths that exit through one or other of the two sides of $\tilde{\gamma}$. We say that D is *harmonically symmetric in γ with respect to z_0* if, for each subarc $\tilde{\gamma}$ of γ , Brownian motion starting from z_0 has equal probability of exiting D from either side of $\tilde{\gamma}$. Now we can state the extension of Jenkins' condition from [9].

Theorem. *Suppose that D is an extremal domain for the univalent Bloch-Landau constant. Suppose that γ is an internal boundary arc for D , no point of which lies on the boundary of an extremal disk. Then D is harmonically symmetric in γ with respect to 0.*

Jenkins' original condition [14] corresponds to the case when γ is a straight line segment, in which case he concludes that γ lies on a ray through the origin and that

the extremal domain D is symmetric in this ray. It is shown in [9] that harmonic symmetry and geometric symmetry coincide when the curve γ is rectilinear, so that the theorem stated above is indeed an extension of Jenkins' condition.

Our work in [3] led us to believe that, for domains of prescribed inradius, the extremal domains for the expected lifetime of Brownian motion for the first Dirichlet eigenvalue (presuming that extremal domains exist) and for the univalent Bloch-Landau constant are the same. While it is still not clear whether this is the case or not, we conjecture herein that extremal domains for the expected lifetime of Brownian motion must satisfy the extension of Jenkins' condition mentioned above and that, as in the Bloch-Landau problem, these domains always contain an extremal disk.

Conjecture. *Suppose that D is an extremal domain for the expected lifetime of Brownian motion in that $E_{z_0}\tau_D = AR_D^2$, where $z_0 \in D$. Suppose that γ is an internal boundary arc for D , no point of which lies on the boundary of an extremal disk. Then D is harmonically symmetric in γ with respect to z_0 . Furthermore, an extremal domain will contain an extremal disk, that is a disk of radius R_D .*

The problem of deciding whether or not there are extremal domains for the first Dirichlet eigenvalue of the Laplacian is unresolved at the time of writing, and appears not to yield to the techniques herein. It would certainly be interesting to have an answer to this question. In [7] it is shown that the supremum of the torsion function is bounded in terms of the inverse of the bottom of the spectrum of the Laplacian, which is in the spirit of the present paper. Moreover, this latter result holds in all dimensions.

2. Convergence of domains and of the expected exit time

The proof of Theorem 1 relies on Proposition 1 below, which states, loosely speaking, that the convergence of a sequence of simply connected domains of uniformly bounded inradius implies the convergence of the expected lifetime of Brownian motion for these domains.

In this section, $\{f_n\}_1^\infty$ will always stand for a sequence of conformal mappings of the unit disk D , normalised so that $f_n(0) = 0$ and $f'_n(0) > 0$. If $\{f_n\}_1^\infty$ converges uniformly on compact subsets of D , then the limit function f is either constant or is itself a conformal mapping of D . We write D_n for $f_n(D)$, $n \geq 1$, and D for $f(D)$.

Proposition 1. *Suppose that the sequence of conformal maps $\{f_n\}_1^\infty$ converges to a non-constant function f , uniformly on compact subsets of D . Suppose that*

$$M = \sup_n R_{D_n} < \infty. \quad (2.1)$$

Then

$$\lim_{n \rightarrow \infty} E_0\tau_{D_n} = E_0\tau_D. \quad (2.2)$$

2.1. Kernel convergence

Carathéodory described the analytic concept of convergence of a sequence of conformal mappings, uniformly on compact subsets of the unit disk, in terms of the geometric concept of ‘kernel convergence’ of the corresponding sequence of image domains. We describe this briefly here, referring to [10, chapter 15] for a complete treatment. Though kernel convergence is not needed for the proof of Proposition 1, the geometric meaning of the convergence of a sequence of conformal maps sheds light on its statement and is necessary for the construction of the examples below.

The *pre-kernel* of a sequence of domains $\{\Omega_n\}_1^\infty$ (not necessarily simply connected, but each of which contains 0) is defined to be the set of all points w for which there is some positive number r so that the closed disk $\overline{D(w, r)}$ is contained in Ω_n for all sufficiently large n . If 0 lies in the pre-kernel, we define the *kernel* of $\{\Omega_n\}_1^\infty$ to be the component of the pre-kernel of $\{\Omega_n\}_1^\infty$ that contains 0. Equivalently, if 0 lies in the pre-kernel, the kernel of $\{\Omega_n\}_1^\infty$ is the largest domain that contains 0 with the property that each compact subset of Ω is contained in Ω_n for all sufficiently large n . Finally, a sequence of domains $\{\Omega_n\}_1^\infty$ is said to converge to the domain Ω in the sense of Carathéodory if 0 lies in the pre-kernel of $\{\Omega_n\}_1^\infty$ and Ω is the kernel of each subsequence $\{\Omega_{n_k}\}_1^\infty$ of the sequence of domains $\{\Omega_n\}_1^\infty$.

The Carathéodory kernel theorem states that the sequence of conformal mappings $\{f_n\}_1^\infty$ (as normalised above) converges to a conformal mapping f uniformly on compact subsets of \mathbb{D} if and only if $\{D_n\}_1^\infty$ converges to D in the sense of Carathéodory and D is not the whole complex plane.

Before moving to the proof of Proposition 1 we provide two pertinent examples. While convergence of domains in the sense of Carathéodory, together with (2.1), is sufficient to guarantee convergence of the expected lifetimes in the case of simply connected domains, this is not true in general as the first example makes clear. The second example shows that condition (2.1) cannot be dropped, even if the domains D_n are simply connected.

Example. Set

$$\Omega = \{z: -1 < \operatorname{Re} z < 2, |\operatorname{Im} z| < 1\}.$$

We choose a countable, dense set of points $\{w_n\}_{n=1}^\infty$ on the line segment $(1-i, 1+i)$ and set

$$\Omega_n = \Omega \setminus \{w_1, w_2, \dots, w_n\}.$$

The pre-kernel of $\{\Omega_n\}_1^\infty$ is the union of the sets $\{-1 < \operatorname{Re} z < 1, |\operatorname{Im} z| < 1\}$ and $\{1 < \operatorname{Re} z < 2, |\operatorname{Im} z| < 1\}$, so that the kernel of $\{\Omega_n\}_1^\infty$ is

$$\tilde{\Omega} = \{z: -1 < \operatorname{Re} z < 1, |\operatorname{Im} z| < 1\}.$$

Moreover, $\tilde{\Omega}$ is the kernel of any subsequence of $\{\Omega_n\}_1^\infty$, so that $\{\Omega_n\}_1^\infty$ converges to $\tilde{\Omega}$ in the sense of Carathéodory. However, $E_0\tau_{\Omega_n} = E_0\tau_\Omega$ for each n so that

$$\lim_{n \rightarrow \infty} E_0\tau_{\Omega_n} = E_0\tau_\Omega > E_0\tau_{\tilde{\Omega}},$$

and (2.2) does not hold.

Example. We write H for the half-plane $\{z : \operatorname{Re} z > 1\}$ and set

$$D_n = D(0, 1 + 1/n) \cup H.$$

Then $\{D_n\}_{n=1}^\infty$ converges to the unit disk D in the sense of Carathéodory, so that $\mathbf{E}_0\tau_D = 1/2$. On the other hand, if $z \in H$, $\mathbf{E}_z\tau_{D_n} \geq \mathbf{E}_z\tau_H = \infty$. We will deduce from this that

$$\mathbf{E}_0\tau_{D_n} = \infty, \quad \text{for each } n.$$

The expected lifetime can be written as

$$\mathbf{E}_z\tau_\Omega = \int_\Omega G(z, w; \Omega) dw \quad (2.3)$$

in terms of the Green function $G(z, w; \Omega)$ of a general domain Ω with pole at $z \in \Omega$ (see, for example, [3, identity (1.1)]). Regarding the Green function $G(z, w; \Omega)$ as a harmonic function of z , it is a consequence of the maximum principle that, for fixed z_1 and z_2 in Ω , there is a constant C independent of w , once w is near neither z_1 nor z_2 , such that

$$\frac{1}{C} \leq \frac{G(z_1, w; \Omega)}{G(z_2, w; \Omega)} \leq C.$$

It follows from (2.3) that $\mathbf{E}_{z_1}\tau_\Omega = \infty$ if and only if $\mathbf{E}_{z_2}\tau_\Omega = \infty$. This also follows from the strong Markov property of the Brownian motion. In our case, we deduce from $\mathbf{E}_z\tau_{D_n} = \infty$, $z \in H \subset D_n$, that $\mathbf{E}_0\tau_{D_n} = \infty$. Thus (2.2) fails for this sequence of domains since $\infty = \lim_{n \rightarrow \infty} \mathbf{E}_0\tau_{D_n} \neq \mathbf{E}_0\tau_D = 1/2$. The condition (2.1) is therefore necessary: it fails to hold in this example as each domain D_n has infinite inradius.

2.2. Proof of Proposition 1

The existence of the limit in (2.2) needs to be established and it needs to be shown that it equals $\mathbf{E}_0\tau_D$.

In the case $z = 0$ and Ω simply connected, (2.3) becomes

$$\mathbf{E}_0\tau_\Omega = \frac{1}{\pi} \int_D \log \frac{1}{|z|} |h'(z)|^2 dz, \quad (2.4)$$

in terms of a conformal mapping h from D onto Ω with $h(0) = 0$, (see [3, Lemma 1]). Thus, if we set

$$g_n(z) = \frac{1}{\pi} \log \frac{1}{|z|} |f'_n(z)|^2 \quad \text{and} \quad g(z) = \frac{1}{\pi} \log \frac{1}{|z|} |f'(z)|^2,$$

then

$$\mathbf{E}_0\tau_{D_n} = \int_D g_n(z) dz \quad \text{and} \quad \mathbf{E}_0\tau_D = \int_D g(z) dz.$$

Moreover, since $f_n \rightarrow f$ (uniformly on compact subsets of D), $g_n \rightarrow g$ in D (uni-

formly on compact subsets) and an application of Fatou's Lemma yields

$$\liminf_{n \rightarrow \infty} \mathbf{E}_0 \tau_{D_n} \geq \int_{\mathbb{D}} \left(\liminf_{n \rightarrow \infty} g_n(z) \right) dz = \int_{\mathbb{D}} g(z) dz = \mathbf{E}_0 \tau_D. \quad (2.5)$$

In order to obtain an estimate in the other direction, we use (2.1). As has often been noted in the past, it follows from the Koebe $\frac{1}{4}$ -Theorem and (2.1) that, for each n and for $z \in \mathbb{D}$,

$$|f'_n(z)| \leq 4 \frac{\text{dist}(f_n(z), \partial D_n)}{1 - |z|^2} \leq \frac{4M}{1 - |z|^2} \leq \frac{4M}{1 - |z|}.$$

In essence, this demonstrates the well-known fact that a conformal mapping belongs to the Bloch space if and only if the image of the unit disk under the map has finite inradius. Integrating along the radius from 0 to z , and using $f_n(0) = 0$, yields

$$|f_n(z)| \leq 4M \int_0^{|z|} \frac{d\rho}{1 - \rho} = 4M \log \left(\frac{1}{1 - |z|} \right).$$

As a consequence, the image of the annulus

$$A_k = \{z: 1 - 2^{-k+1} \leq |z| \leq 1 - 2^{-k}\}$$

under f_n lies inside the disk centre 0 and radius $(4M \log 2)k$. Thus, for each natural number k_0 and each n ,

$$\begin{aligned} \int_{\{1-2^{-k_0} < |z| < 1\}} g_n(z) dz &= \sum_{k=k_0+1}^{\infty} \int_{A_k} \left(\log \frac{1}{|z|} \right) |f'_n(z)|^2 dz \\ &\leq 2 \sum_{k=k_0+1}^{\infty} \int_{A_k} (1 - |z|) |f'_n(z)|^2 dz \\ &\leq 2 \sum_{k=k_0+1}^{\infty} \frac{1}{2^{k-1}} \text{area}(f_n(A_k)) \\ &\leq (64\pi M^2 \log^2 2) \sum_{k=k_0+1}^{\infty} \frac{k^2}{2^k}, \end{aligned} \quad (2.6)$$

which is $o(1)$ as $k_0 \rightarrow \infty$.

We may now obtain a lower bound for $\mathbf{E}_0 \tau_D$. Given ϵ positive we may, by what we have just proved, choose r less than 1 so that

$$\sup_n \left(\int_{\{r < |z| < 1\}} g_n(z) dz \right) \leq \epsilon.$$

Using the fact that g_n converges uniformly to g on the disk $D(0, r)$, we then find

that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \mathbf{E}_0 \tau_{D_n} &= \limsup_{n \rightarrow \infty} \left[\int_{D(0,r)} g_n(z) dz + \int_{\{r < |z| < 1\}} g_n(z) dz \right] \\
&\leq \limsup_{n \rightarrow \infty} \left[\int_{D(0,r)} g_n(z) dz \right] + \limsup_{n \rightarrow \infty} \left[\int_{\{r < |z| < 1\}} g_n(z) dz \right] \\
&\leq \int_{D(0,r)} g(z) dz + \epsilon \\
&\leq \mathbf{E}_0 \tau_D + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we deduce that $\limsup_{n \rightarrow \infty} \mathbf{E}_0 \tau_{D_n} \leq \mathbf{E}_0 \tau_D$. Putting this together with (2.6), we arrive at

$$\limsup_{n \rightarrow \infty} \mathbf{E}_0 \tau_{D_n} \leq \mathbf{E}_0 \tau_D \leq \liminf_{n \rightarrow \infty} \mathbf{E}_0 \tau_{D_n},$$

which proves (2.2).

2.3. A lower bound for the hyperbolic metric in terms of the expected lifetime

One last ingredient is needed to complete the preparations for the proof of Theorem 1.

Lemma 1. *Let h be a conformal mapping of the unit disk \mathbb{D} with $h(0) = 0$. Let $R_{h(\mathbb{D})} \leq M < \infty$ and $\mathbf{E}_0 \tau_{h(\mathbb{D})} \geq L > 0$. There is a positive number η , depending only on M and L , such that*

$$|h'(0)| \geq \eta.$$

Note that this lemma is demonstrably false if it is not assumed that $h(\mathbb{D})$ has finite inradius. In fact, $h(z) = \eta z / (1 - z)^2$ conformally maps the unit disk onto the plane slit along the negative real axis from $-\infty$ to $-\eta/4$. Even though $\mathbf{E}_0 \tau_{h(\mathbb{D})} = \infty$ in each case, $h'(0) = \eta$ can be as small as one wishes.

Proof of Lemma 1 We again use (2.4), so that

$$\mathbf{E}_0 \tau_{h(\mathbb{D})} = \int_{\mathbb{D}} \frac{1}{\pi} \log \frac{1}{|z|} |h'(z)|^2 dz. \quad (2.7)$$

Since $h(\mathbb{D})$ has finite inradius, it follows from (2.6) that there is an $r \in (0, 1)$, that depends only on M and L , such that

$$\int_{\{r < |z| < 1\}} \frac{1}{\pi} \log \frac{1}{|z|} |h'(z)|^2 dz \leq \frac{1}{2} L. \quad (2.8)$$

We may assume that $r \geq e^{-1/2}$, so that $\log(1/r) \leq 1/2$.

Let $h(z) = \sum_{n=1}^{\infty} a_n z^n$ be the power series expansion for h in D . Then, by (2.7) and (2.8),

$$\begin{aligned} \frac{1}{2} \mathbf{E}_0 \tau_{h(D)} &\leq \int_{D(0,r)} \frac{1}{\pi} \log \frac{1}{|z|} |h'(z)|^2 dz \\ &= \sum_{n=1}^{\infty} \left(n \log \frac{1}{r} + \frac{1}{2} \right) r^{2n} |a_n|^2 \\ &\leq \sum_{n=1}^{\infty} n r^{2n} |a_n|^2 \\ &= \frac{1}{\pi} \text{area} (h(D(0, r))). \end{aligned} \quad (2.9)$$

By the classical distortion theorems for univalent functions [11, p. 33]

$$|h(z)| \leq |h'(0)| \frac{r}{(1-r)^2}, \text{ for } |z| \leq r.$$

It now follows from (2.9) that

$$\frac{1}{2} L \leq \frac{1}{2} \mathbf{E}_0 \tau_{h(D)} \leq \frac{1}{\pi} \text{area} (h(D(0, r))) \leq |h'(0)|^2 \frac{r^2}{(1-r)^4}.$$

Hence

$$|h'(0)| \geq \sqrt{\frac{L}{2}} \frac{(1-r)^2}{r},$$

which proves the lemma.

2.4. Proof of Theorem 1

By definition of \mathcal{A} as the supremum of the quantity $\mathbf{E}_z \tau_D / R_D^2$ over all simply connected domains D and over all points z in D , we may choose a sequence of simply connected domains D_n , and a point z_n in each D_n , such that

$$\frac{\mathbf{E}_{z_n} \tau_{D_n}}{R_{D_n}^2} \geq \mathcal{A} - \frac{1}{n}, \quad n \geq 1.$$

By applying a suitable translation, we may assume that $z_n = 0$ in each case and then, by applying a suitable scaling, we may further assume that $\mathbf{E}_0 \tau_{D_n} = 1$. We write f_n for the conformal mapping of the unit disk D onto D_n for which $f_n(0) = 0$ and $f'_n(0) > 0$. Then, for $n \geq 1$,

$$\frac{1}{R_{f_n(D)}^2} \geq \mathcal{A} - \frac{1}{n}, \text{ that is } R_{f_n(D)} \leq \frac{1}{\sqrt{\mathcal{A} - 1/n}}. \quad (2.10)$$

Thus, since $\mathcal{A} \geq 1.584$, (2.1) is satisfied by the sequence of domains $\{D_n\}$, with $M = 1/\sqrt{0.584}$.

Another expression for the expected lifetime $\mathbf{E}_0 \tau_{h(D)}$ is $\frac{1}{2} \sum_{n=1}^{\infty} |a_n|^2$, where h

is conformal in the unit disk and $h(z) = \sum_{n=1}^{\infty} a_n z^n$ (see [3, lemma 1.1]). Since $E_0 \tau_{f_n(\mathbb{D})} = 1$, we deduce that $f'_n(0) \leq \sqrt{2}$. The classical distortion theorems for univalent functions therefore apply and yield that the sequence of functions $\{f_n\}$ is uniformly bounded on compact subsets of the unit disk. Some subsequence of $\{f_n\}_1^{\infty}$ (which we again refer to as $\{f_n\}_1^{\infty}$) therefore converges uniformly on compact subsets of \mathbb{D} to an analytic function f , which is either constant or univalent.

By Lemma 1, there is a positive η for which $f'_n(0) \geq \eta$ for each n . Hence $f'(0) \geq \eta$ and is therefore positive, so that f is not constant. We may therefore assume that the sequence of conformal maps $\{f_n\}_1^{\infty}$ converges uniformly on compact subsets of the unit disk to a conformal mapping f of the unit disk onto a simply connected domain D . By Proposition 1, $E_0 \tau_D = 1$. It remains to show that $1/R_{f(\mathbb{D})}^2 = \mathcal{A}$, which is where we use an argument of Robinson from [19].

By definition of \mathcal{A} , $1/R_{f(\mathbb{D})}^2 \leq \mathcal{A}$. Suppose, if possible, that $1/R_{f(\mathbb{D})}^2 < \mathcal{A}$, so that $R_{f(\mathbb{D})} > 1/\sqrt{\mathcal{A}}$. There is, then, a disk $D(a, r)$ contained in D with radius $r = 1/\sqrt{\mathcal{A}} + 3\epsilon$ for some positive ϵ . It follows that the closed disk $\overline{D(a, r - \epsilon)}$ is contained in D . We denote by γ the simple closed curve in \mathbb{D} which is the pre-image under f of the circle $C(a, r - \epsilon)$. Then $f_n \rightarrow f$ uniformly on γ . In particular, there is a natural number N such that $|f_n - f| \leq \epsilon$ on γ for $n \geq N$. But then $|f_n - a| \geq r - 2\epsilon = 1/\sqrt{\mathcal{A}} + \epsilon$ on γ , so that $f_n(\gamma)$ encloses the disk $D(a, 1/\sqrt{\mathcal{A}} + \epsilon)$. We have found that $R_{f_n(\mathbb{D})} \geq 1/\sqrt{\mathcal{A}} + \epsilon$, which violates (2.10) for sufficiently large n . Hence D is an extremal domain for the inequality (1.1), proving Theorem 1.

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