WEIGHTED COMPOSITION OPERATORS BETWEEN WEIGHTED BLOCH TYPE SPACES

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Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\phi : \mathbb{D} \to \mathbb{D}$ as well as $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps. For a holomorphic function f on $\mathbb{D}$ the weighted composition operator $C_{\phi,\psi}$ is defined by $(C_{\phi,\psi} f)(z) = \psi(z) f(\phi(z))$ for every $z \in \mathbb{D}$. We characterize when weighted composition operators acting between weighted Bloch type spaces are bounded resp. compact.

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1. Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and ψ be an analytic map on $\mathbb{D}$. Such maps induce the so-called *weighted composition operator*

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi(f \circ \phi),$$

where $H(\mathbb{D})$ denotes the set of all holomorphic functions on $\mathbb{D}$. In case that $\psi(z) = 1$ for every $z \in \mathbb{D}$ we simply write C_ϕ and obtain the classical *composition operator*. Such operators have been investigated on various spaces of holomorphic functions and by several authors, see e.g. [17], [8], [13], [4], [6], [5], [11].
In this paper we are interested in operators $C_{\phi,\psi}$ acting between different weighted Bloch type spaces. For a continuous, strictly positive and bounded function (*weight*) on $\mathbb{D}$ we say that the *weighted Bloch type space* B_v is the collection of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_v} := \sup_{z \in \mathbb{D}} v(z)|f'(z)| < \infty.$$

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Provided, we identify functions that differ by a constant, $\|.\|_{B_v}$ becomes a norm and B_v a Banach space. Moreover, the "little weighted Bloch type space" B_v^0 is defined by

$$B_v^0 := \{f \in B_v;\ \lim_{|z|\to 1} v(z)|f'(z)| = 0\}.$$

In [17] Ohno, Stroethoff and Zhao characterized boundedness and compactness of weighted composition operators in the framework of weighted Bloch type spaces generated by the standard weights, i.e. weights of the form $v(z) = (1-|z|^2)^\alpha$ with $\alpha \in \mathbb{R}$ and $\alpha > 0$.
We characterize these properties of $C_{\phi,\psi}$ acting between different more general Bloch type spaces. At this point it should be mentioned that in [14] Montes-Rodríguez gave a formula for the essential norm and hence a characterization for the compactness of a non-weighted composition operator on the classical Bloch spaces. It will turn out that his result is a special case.
Moreover, we would like also to refer to the articles [9] and [10] for recent work in the setting of Bloch spaces on the polydisk.

2. Notation, auxiliary results and basic facts

First, for the notation as well as a general introduction to the concept of (weighted) composition operators, we refer the reader to the excellent monographs [18] and [7]. In order to address the problem mentioned above we will also need the so called *weighted Banach spaces of holomorphic functions*. Thus, for a weight v we consider

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\} \text{ and}$$
$$H_v^0 := \{f \in H_v^\infty;\ \forall K \subset\subset \mathbb{D}\ \exists\varepsilon > 0:\ v(z)|f(z)| < \varepsilon\ \forall z \in \mathbb{D}\backslash K\}.$$

Endowed with norm $\|.\|_v$ these are Banach spaces. Such spaces occur naturally in several problems regarding the growth conditions of analytic functions. This means they arise in complex analysis, spectral theory, Fourier analysis, partial differential equations and convolution equations. They were intensively studied e.g. in [3] and [2].
The study of weighted spaces resp. of operators acting on weighted spaces requires the concept of the so-called *associated weight*, which was introduced by Anderson and Duncan in [1] and thoroughly investigated by Bierstedt, Bonet and Taskinen in [3].
Given a weight v, its associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \le 1\}} \text{ for every } z \in \mathbb{D}.$$

Bierstedt, Bonet and Taskinen showed the following very useful facts concerning associated weights:

(a) $0 < v \le \tilde{v}$,
(b) $\tilde{v}$ is continuous,

(c) for every $z \in \mathbb{D}$ there exists a function $f_z \in H_v^\infty$ with $\|f_z\|_v \leq 1$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.

(d) H_v^∞ and $H_{\tilde{v}}^\infty$ are isometrically isomorphic.

In the following we are mainly interested in *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. If such weights satisfy additionally $\lim_{|z|\to 1} v(z) = 0$ they are called *typical*. Since, in general it is quite difficult to compute the associated weight, we need some conditions, when v and $\tilde{v}$ are equivalent, i.e. when there exists a constant $k > 0$ such that

$$v(z) \leq \tilde{v}(z) \leq kv(z) \text{ for every } z \in \mathbb{D}.$$

We say that a weight v is *essential*, if v and $\tilde{v}$ are equivalent. Bonet, Domański and Lindström proved that a radial weight which satisfies condition

$$(L1) \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

(which is due to Lusky, see [12]) is an essential weight. Examples of such weights include among others the standard weights $v_p(z) = (1-|z|^2)^p$, $p > 0$, as well as the logarithmic weights $w_q(z) = (1-\log(1-|z|^2))^q$, $q < 0$. Recall that by [6] weighted composition operators $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ are bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} < \infty.$$

If we assume additionally that v and w are typical weights, then $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is compact if and only if $\limsup_{|\phi(z)|\to 1} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = 0$.

The investigation of the compactness of the operator $C_{\phi,\psi} : B_v \to B_w$ requires the following result which can be easily derived from the result of Cowen and MacCluer, see [7] Proposition 3.11.

Proposition 2.1 (Cowen-MacCluer, [7] Proposition 3.11)). *$C_{\phi,\psi} : B_v \to B_w$ resp. $C_\phi : B_v \to B_w$ is compact if and only if whenever $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in B_v such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi,\psi} f_n \to 0$ resp. $C_\phi f_n \to 0$ in B_w.*

We need the following characterization of the boundedness resp. compactness of operators C_ϕ acting between weighted Bloch type spaces, which in case of standard weights can be easily derived from the boundedness resp. compactness conditions of weighted composition operators acting between weighted Banach spaces of holomorphic functions. We will analyze a more general framework here.

Proposition 2.2. *Let v and w be weights. Then the composition operator $C_\phi : B_v \to B_w$ is bounded if and only if $\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))} < \infty$.*

Proof. First, we suppose that $C_\phi : B_v \to B_w$ is bounded. We assume to the contrary that we can find a sequence $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that $\frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))} \geq n$ for every $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$ we can choose $f_n \in B_v$ with $\|f_n\|_{B_v} \leq 1$ and $|f_n'(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$. Then, by the boundedness of the operator $C_\phi : B_v \to B_w$ we can find a constant $c > 0$ such that

$$c \geq w(z_n)|f_n'(\phi(z_n))||\phi'(z_n)| = \frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))} \geq n \text{ for every } n \in \mathbb{N},$$

which is a contradiction.
Conversely, we get that

$$\begin{aligned}\|C_\phi f\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f'(\phi(z))| &\leq \sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))}\tilde{v}(\phi(z))|f'(\phi(z))| \\ &\leq \sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))}\|f\|_{B_{\tilde{v}}}\end{aligned}$$

and the claim follows. ■

In the sequel, we assume the following setting. Let ν be a holomorphic function on $\mathbb{D}$ such that ν is non-vanishing, strictly positive and decreasing on $[0, 1)$. Furthermore we suppose that $\lim_{|z|\to 1} \nu(z) = 0$ and that ν' is bounded on $\mathbb{D}$. Then we define the corresponding weight by

$$v(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}. \tag{2.1}$$

Let us give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1 - z)^\alpha$, $\alpha \geq 1$, for every $z \in \mathbb{D}$. Then the corresponding weight is given by $v(z) = (1 - |z|^2)^\alpha$ for every $z \in \mathbb{D}$.

(ii) For $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$, for every $z \in \mathbb{D}$, we obtain $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$ for every $z \in \mathbb{D}$.

(iii) Choose $\nu(z) = \sin(1 - z)$ for every $z \in \mathbb{D}$ and get $v(z) = \sin(1 - |z|^2)$ for every $z \in \mathbb{D}$.

Next, we fix a point $p \in \mathbb{D}$ and introduce a function

$$v_p(z) := \nu(\overline{\phi(p)}z) \text{ for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, so is the function v_p. Furthermore, $v_p(\phi(p)) = \nu(|\phi(p)|^2) = v(\phi(p))$ and $v_p'(z) = \overline{\phi(p)}\nu'(\overline{\phi(p)}z)$ for every $z \in \mathbb{D}$, i.e. $v_p'(\phi(p)) = \overline{\phi(p)}\nu'(|\phi(p)|^2)$. Moreover, in particular we assume that there is a constant $M > 0$ with

$$\sup_{p\in\mathbb{D}}\sup_{z\in\mathbb{D}} \frac{v(z)}{|v_p(z)|} \leq M. \tag{2.2}$$

Here are some examples satisfying (2.2):

(a) Consider $v(z) = 1 - |z|^2$ for every $z \in \mathbb{D}$. Then

$$\frac{v(z)}{|v_p(z)|} = \frac{1-|z|^2}{|1-\overline{\phi(p)}z|} \leq \frac{1-|z|^2}{1-|z|} \leq 1 + |z| \leq 2$$

for every $z \in \mathbb{D}$.

(b) We study $v(z) = \frac{1}{1-\log(1-|z|^2)}$ for every $z \in \mathbb{D}$. This weight has the desired property since $|1-\log(1-\overline{\phi(p)}z)| \leq 1-\log(1-|z|)$ for every $z \in \mathbb{D}$ and the function $\frac{1-\log(1-|z|)}{1-\log(1-|z|^2)}$ is continuous and tends to 1 if $|z| \to 1$.

3. Boundedness

In this section we will investigate when the operator $C_{\phi,\psi} : B_v \to B_w$ is bounded. We will use two different methods to approach this problem. Let us first state a simple lemma which will be very useful.

Lemma 3.1. *For every $f \in B_v$ we have that*

$$|f(z) - f(0)| \leq \|f\|_{B_v} |V(|z|^2) - V(0)|,$$

where V is a holomorphic function with $V'(z) = \frac{1}{\nu(z)}$ for every $z \in \mathbb{D}$.

PROOF. Fix $z \in \mathbb{D}$. We have that

$$\begin{aligned} |f(z) - f(0)| &\leq |z| \int_0^1 |f'(tz)|\, dt \leq |z| \|f\|_{B_v} \int_0^1 \frac{1}{v(tz)}\, dt \\ &\leq \|f\|_{B_v} \int_0^1 \frac{1}{\nu(t^2|z|^2)}\, dt \leq \|f\|_{B_v} \int_0^1 \frac{1}{\nu(t|z|^2)}\, dt \\ &= \|f\|_{B_v} |V(|z|^2) - V(0)|. \end{aligned}$$

■

Theorem 3.2. *Let w be a weight and v be a weight as defined in (2.1). Moreover, let V be a holomorphic function on $\mathbb{D}$ such that $V'(z) = \frac{1}{\nu(z)}$ for every $z \in \mathbb{D}$ and $L := \sup_{p\in\mathbb{D}} \sup_{z\in\mathbb{D}} \left| \frac{V(\overline{\phi(p)}z)}{V(|\phi(p)|^2)} \right| < \infty$. Then the operator $C_{\phi,\psi} : B_v \to B_w$ is bounded if and only if the following conditions hold:*

(a) $\psi \in B_w$,

(b) $\sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))} |\psi(z)||\phi'(z)| < \infty$,

(c) $\sup_{z\in\mathbb{D}} w(z)|\psi'(z)||V(|\phi(z)|^2)| < \infty$.

Proof. First, let us assume that conditions (a)-(c) hold. An application of Lemma 3.1 yields for $f \in B_v$

$$\begin{aligned}
\sup_{z\in\mathbb{D}} w(z)|(C_{\phi,\psi}f)'(z)| &= \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f(\phi(z)) + \psi(z)f'(\phi(z))\phi'(z)| \\
&\leq \sup_{z\in\mathbb{D}} w(z)|\psi'(z)||f(\phi(z)) - f(0)| \\
&+ \sup_{z\in\mathbb{D}} w(z)|\psi'(z)||f(0)| + \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z)||\psi(z)|\|f\|_{B_v} \\
&\leq \sup_{z\in\mathbb{D}} w(z)|\psi'(z)||V(|\phi(z)|^2) - V(0)|\|f\|_{B_v} \\
&+ \sup_{z\in\mathbb{D}} \frac{w(z)}{v(0)}|\psi'(z)|\|f\|_{B_v} + \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z)||\psi(z)|\|f\|_{B_v}.
\end{aligned}$$

Hence the operator must be bounded.
Next, we suppose that $C_{\phi,\psi} : B_v \to B_w$ is bounded. Take $f(z) = 1$ for every $z \in \mathbb{D}$. Then, by the boundedness of the operator, we obtain

$$\sup_{z\in\mathbb{D}} w(z)|\psi'(z)| < \infty, \text{ thus } \psi \in B_w, \tag{3.1}$$

and condition (a) follows.
It remains to show (b) and (c). We start with proving (c). To do this we fix $p \in \mathbb{D}$ and select

$$f_p(z) = 2V(\overline{\phi(p)}z) - \frac{V(\overline{\phi(p)}z)^2}{V(|\phi(p)|^2)} \text{ for every } z \in \mathbb{D}.$$

Then

$$f_p'(z) = \frac{2\overline{\phi(p)}}{\nu(\overline{\phi(p)}z)} - \frac{2\overline{\phi(p)}V(\overline{\phi(p)}z)}{\nu(\overline{\phi(p)}z)V(|\phi(p)|^2)} \text{ for every } z \in \mathbb{D}.$$

Moreover, $\|f_p\|_{B_v} \leq 2\sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(p)}z)|} + 2\sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(p)}z)|} \left|\frac{V(\overline{\phi(p)}z)}{V(|\phi(p)|^2)}\right| \leq 2M(1+L)$.
Furthermore, $f_p(\phi(p)) = V(|\phi(p)|^2)$ as well as $f_p'(\phi(p)) = 0$ and thus

$$2M(1+L)\|C_{\phi,\psi}\| \geq \|C_{\phi,\psi}f_p\|_{B_w} \geq w(p)|\psi'(p)|V(|\phi(p)|^2).$$

Hence condition (c) follows.

Next, take the function

$$g_p(z) := V(\overline{\phi(p)}z) \text{ for every } z \in \mathbb{D}$$

with

$$g_p'(z) = \frac{1}{\nu(\overline{\phi(p)}z)} \text{ for every } z \in \mathbb{D}$$

and $\|g_p\|_{B_v} \leq \sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(p)}z)|} \leq M$. Thus the boundedness of the operator $C_{\phi,\psi} : B_v \to B_w$ yields

$$w(p)|C_{\phi,\psi}g_p(p)| = w(p)\left|\psi'(p)V(|\phi(p)|^2) + \frac{\psi(p)}{v(\phi(p))}\right| \leq M\|C_{\phi,\psi}\|.$$

Since $\sup_{z\in\mathbb{D}} w(z)|\psi'(z)V(|\phi(z)|^2)| < \infty$, we get that

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{v(\phi(z))} < \infty,$$

as desired. ■

Theorem 3.3. *Let w be a weight and v be a weight as defined in (2.1) such that there is a holomorphic function V with $V'(z) = \frac{1}{\nu(z)}$ for every $z \in \mathbb{D}$. The operator $C_{\phi,\psi} : B_v \to H_w^\infty$ is bounded if and only if the following conditions hold*

(a) $\psi \in H_w^\infty$.

(b) $\sup_{z\in\mathbb{D}} w(z)|\psi(z)||V(|\phi(z)|^2)| < \infty$.

PROOF. Let us first assume that conditions (a) and (b) hold. For every $f \in B_v$ we obtain by using Lemma 3.1

$$\begin{aligned}
\|C_{\phi,\psi}f\|_w &= \sup_{z\in\mathbb{D}} w(z)|\psi(z)||f(\phi(z))| \\
&\leq \sup_{z\in\mathbb{D}} w(z)|\psi(z)||f(\phi(z)) - f(0)| + \sup_{z\in\mathbb{D}} w(z)|\psi(z)||f(0)| \\
&\leq \sup_{z\in\mathbb{D}} w(z)|\psi(z)||V(|\phi(z)|^2) - V(0)|\|f\|_{B_v} + \sup_{z\in\mathbb{D}} \frac{w(z)}{v(0)}|\psi(z)|\|f\|_{B_v} \\
&\leq \left(\sup_{z\in\mathbb{D}} w(z)|\psi(z)||V(|\phi(z)|^2)| + C\sup_{z\in\mathbb{D}} w(z)|\psi(z)|\right)\|f\|_{B_v}.
\end{aligned}$$

Hence the operator must be bounded. Conversely, first consider the function $f(z) = 1$ for every $z \in \mathbb{D}$. Obviously, $f \in B_v$ and, by the boundedness of the operator $C_{\phi,\psi}$, it immediately follows that $\psi \in H_w^\infty$. In order to show condition (b) we fix $p \in \mathbb{D}$ and select

$$f_p(z) := V(\overline{\phi(p)}z) \text{ for every } z \in \mathbb{D}.$$

Then $f_p'(z) = \frac{\overline{\phi(p)}}{\nu(\phi(p)z)}$ for every $z \in \mathbb{D}$. Moreover $\|f\|_{B_v} = \sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(p)}z)|} \leq M$. Thus

$$M\|C_{\phi,\psi}\| \geq \|C_{\phi,\psi}f_p\|_{B_w} \geq w(p)|\psi(p)||V(|\phi(p)|^2).$$

Hence the claim follows. ■

Theorem 3.4. *Let v and w be weights. If*

(a) $C_{\phi,\psi'} : B_v \to H_w^\infty$ is bounded, and,

(b) $C_\phi : B_v \to B_{|\psi|w}$ is bounded,

then the weighted composition operator $C_{\phi,\psi} : B_v \to B_w$ is bounded. If we assume additionally that v is a weight as defined in (2.1) such that there is a holomorphic function V with $V'(z) = \frac{1}{\nu(z)}$ for every $z \in \mathbb{D}$ and $L := \sup_{p\in\mathbb{D}} \sup_{z\in\mathbb{D}} \left| \frac{V(\overline{\phi(p)}z)}{V(|\phi(p)|^2)} \right| < \infty$, then the converse is also true.

Proof. Let us first assume that (a) and (b) hold. For every $f \in B_v$ the following holds

$$\|C_{\phi,\psi} f\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f(\phi(z)) + \psi(z)\phi'(z)f'(\phi(z))|$$
$$\leq \|C_{\phi,\psi'} f\|_w + \|C_\phi f\|_{B_{|\psi|w}}.$$

Now, if (a) is fulfilled we can find a constant $M_1 > 0$ such that for every $f \in B_v$

$$\|C_{\phi,\psi'} f\|_w \leq \|C_{\phi,\psi'}\| \|f\|_{B_v} \leq M_1 \|f\|_{B_v}.$$

Moreover, if (b) is satisfied, there is a constant $M_2 > 0$ such that

$$\|C_\phi f\|_{B_{|\psi|w}} \leq \|C_\phi\| \|f\|_{B_v} \leq M_2 \|f\|_{B_v}.$$

Finally,

$$\|C_{\phi,\psi} f\|_{B_w} \leq (M_1 + M_2)\|f\|_{B_v}.$$

Conversely, we first prove (b). By Proposition 2.2 we have to show that $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)||\phi'(z)|}{\tilde{v}(\phi(z))} < \infty$. We fix $p \in \mathbb{D}$ and select $f_p \in B_v$ such that $\|f_p\|_{B_v} \leq 1$ and $|f_p'(\phi(p))| = \frac{1}{\tilde{v}(\phi(p))}$. Next, we put

$$g_p(z) := f_p(z) - f_p(\phi(p)) \text{ for every } z \in \mathbb{D}.$$

Obviously, $g_p \in B_v$ with $\|g_p\|_{B_v} \leq 2$ and $g_p(\phi(p)) = 0$ as well as $|g_p'(\phi(p))| = \frac{1}{\tilde{v}(\phi(p))}$. Since $C_{\phi,\psi} : B_v \to B_w$ is bounded and $p \in \mathbb{D}$ arbitrary, we obtain that $\sup_{p\in\mathbb{D}} \frac{w(p)|\psi(p)||\phi'(p)|}{\tilde{v}(\phi(p))} < \infty$. Hence (b) holds.
It remains to show (a). For a fixed $p \in \mathbb{D}$ we consider

$$f_p(z) = 2V(\overline{\phi(p)}z) - \frac{V(\overline{\phi(p)}z)^2}{V(|\phi(p)|^2)} \text{ for every } z \in \mathbb{D}$$

as well as $f(z) = 1$ and, using Theorem 3.3, obtain the claim. ■

Let us give some illustrating examples.

Example 3.5. (a) Consider the weights $v(z) = (1 - |z|^2)^2 = \tilde{v}(z)$ and $w(z) = e^{-\frac{1}{1-|z|^2}}$ as well as the inducing maps $\phi(z) = \frac{z+1}{2}$ and $\psi(z) = 1 - z$ for every $z \in \mathbb{D}$. Obviously, we have that $\psi'(z) = -1$ and hence $\psi \in B_w$. Moroever $\phi'(z) = \frac{1}{2}$ for every $z \in \mathbb{D}$. $V(z) := \frac{1}{1-z}$ for every $z \in \mathbb{D}$ is a holomorphic

function on $\mathbb{D}$ and satisfies $V'(z) = \frac{1}{(1-z)^2}$ for every $z \in \mathbb{D}$. Now, obviously,

$$\sup_{z\in\mathbb{D}} w(z)\frac{|\psi(z)\phi'(z)|}{v(\phi(z))} = \frac{1}{2}\sup_{z\in\mathbb{D}} e^{-\frac{1}{1-|z|^2}}\frac{|1-z|}{1-|\frac{z+1}{2}|^2} < \infty \text{ and} \tag{3.2}$$

$$\sup_{z\in\mathbb{D}} w(z)|\psi'(z)||V(|\phi(z)|^2)| = \sup_{z\in\mathbb{D}} \frac{e^{-\frac{1}{1-|z|^2}}}{1-|\frac{z+1}{2}|^2} < \infty. \tag{3.3}$$

Hence, the operator $C_{\phi,\psi} : B_v \to B_w$ is bounded.

(b) Take now $w(z) = 1-|z|^2 = v(z) = \tilde{v}(z)$ and $\phi(z) = \frac{z+1}{2}$ as well as $\psi(z) = \frac{1}{(1-z)^3}$ for every $z \in \mathbb{D}$. Clearly, ψ is not an element of B_w. Hence the operator $C_{\phi,\psi} : B_v \to B_w$ is not bounded.

4. Compactness

In this section we will characterize when weighted composition operators $C_{\phi,\psi}$ acting between weighted Bloch type spaces are compact.

Theorem 4.1. *Let v and w be weights and $\psi \in B_w^0$. If*

(a) $\lim_{r\to 1}\sup_{|\phi(z)|>r}\frac{w(z)}{v(\phi(z))}|\psi(z)\phi'(z)| = 0$,

(b) $\lim_{r\to 1}\sup_{|\phi(z)|>r} w(z)|\psi'(z)||V(|\phi(z)|^2)| = 0$,

then the weighted composition operator $C_{\phi,\psi} : B_v \to B_w$ is bounded. If we assume additionally that v is a weight as defined in (2.1) such that there is a holomorphic function V with $V'(z) = \frac{1}{\nu(z)}$ for every $z \in \mathbb{D}$, $\lim_{|z|\to 1}\frac{1}{V(|z|^2)} = 0$ and $L^* := \sup_{p\in\mathbb{D}}\sup_{z\in\mathbb{D}}\left|\frac{V(\overline{p}z)}{V(|p|^2)}\right| < \infty$, then the converse is also true.

PROOF. Let us first assume that conditions (a) and (b) hold. Now, let $(f_n)_n \subset B_v$ be a sequence with $\|f_n\|_{B_v} \le 1$ for every $n \in \mathbb{N}$ and such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$. By Proposition 2.1 we have to show that

$$\|C_{\phi,\psi}f_n\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f_n(\phi(z)) + \psi(z)\phi'(z)f_n'(\phi(z))| \to 0$$

if $n \to \infty$. This reduces the problem to prove that

(i) $\sup_{z\in\mathbb{D}} w(z)|\psi(z)f_n'(\phi(z))\phi'(z)| \to 0$ and

(ii) $\sup_{z\in\mathbb{D}} w(z)|\psi'(z)f_n(\phi(z))| \to 0$

if $n \to \infty$. We first treat the term in (i). If $|\phi(z)| \le r < 1$, then

$$w(z)|\psi(z)f_n'(\phi(z))\phi'(z)| \le L_r \sup_{|\phi(z)|\le r}|f_n'(z)|.$$

In case that $|\phi(z)| > r$, we obtain the following estimate

$$w(z)|\psi(z)\phi'(z)f_n'(\phi(z))| \le \frac{w(z)|\psi(z)\phi'(z)|}{v(\phi(z))}.$$

Joining both cases we arrive at

$$\sup_{z\in\mathbb{D}} w(z)|\psi(z)\phi'(z)f_n'(\phi(z))| \le L_r \sup_{|\phi(z)|\le r} |f_n'(z)| + \sup_{|\phi(z)|>r} w(z)\frac{|\psi(z)\phi'(z)|}{v(\phi(z))} \to 0$$

if $n\to\infty$ and $r\to 1$.
It remains to show (ii). If $|\phi(z)|\le r<1$, we obtain

$$w(z)|\psi'(z)||f_n(\phi(z))| \le K_r \sup_{|\phi(z)|\le r} |f_n(z)|.$$

If $|\phi(z)|>r$, then with Lemma 3.1

$$\begin{aligned} w(z)|\psi'(z)||f_n(\phi(z))| &\le w(z)|\psi'(z)||f_n(\phi(z))-f_n(0)| + w(z)|\psi(z)||f_n(0)| \\ &\le w(z)|\psi'(z)||V(|\phi(z)|^2)| + C|\psi'(z)|w(z). \end{aligned}$$

Again joining both cases we can conclude that (ii) is true.

Conversely, we start with showing (b). Let $(z_n)_n \subset \mathbb{D}$ be such that $|\phi(z_n)|\to 1$ if $n\to\infty$. We put

$$g_n(z) := \frac{3}{V(|\phi(z_n)|^2)} V(\overline{\phi(z_n)}z)^2 - \frac{2}{V(|\phi(z_n)|^2)^2} V(\overline{\phi(z_n)}z)^3$$

for every $z\in\mathbb{D}$ and every $n\in\mathbb{N}$. Then

$$g_n'(z) = \frac{6\overline{\phi(z_n)}}{\nu(\overline{\phi(z_n)}z)} \frac{1}{V(|\phi(z_n)|^2)} \left(V(\overline{\phi(z_n)}z - \frac{1}{V(|\phi(z_n)|^2)} V(\overline{\phi(z_n)}z)^2 \right).$$

Hence $g_n\to 0$ uniformly on the compact subsets of $\mathbb{D}$ and

$$\|g_n\|_{B_v} \le 6\left(\sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(z_n)}z)|} \frac{|V(\overline{\phi(z_n)}z)|}{|V(|\phi(z_n)|^2)|} + \sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu(\overline{\phi(z_n)}z)|} \frac{|V(\overline{\phi(z_n)}z)^2|}{|V(|\phi(z_n)|^2)^2|} \right) \le 6ML^*(1+L^*).$$

Moreover, $g_n'(\phi(z_n))=0$ and $g_n(\phi(z_n)) = V(|\phi(z_n)|^2)$. This yields that

$$w(z_n)|\psi'(z_n)||V(|\phi(z_n)|^2)| \le \|C_{\phi,\psi} g_n\| \to 0 \text{ if } n\to\infty.$$

Thus, (b) follows, and it remains to show (a). Again we consider a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)|\to 1$ if $n\to\infty$ and put

$$h_n(z) := \frac{V(\overline{\phi(z_n)}z)^2}{V(|\phi(z_n)|^2)} \text{ for every } z\in\mathbb{D} \text{ and every } n\in\mathbb{N}.$$

Then $h_n'(z) = \frac{2V(\overline{\phi(z_n)}z)\overline{\phi(z_n)}}{\nu(\overline{\phi(z_n)}z)V(|\phi(z_n)|^2)}$ for every $z\in\mathbb{D}$. Hence $h_n(\phi(z_n)) = V(|\phi(z_n)|^2)$ and $h_n'(\phi(z_n)) = \frac{2\overline{\phi(z_n)}}{v(\overline{\phi(z_n)})}$. Using the compactness of the operator as well as Proposition 2.1 and condition (b), the claim follows. ■

As a corollary we obtain a result due to Montes-Rodríguez.

Corollary 4.2 (Montes-Rodríguez, [14]). *Let $v = w = 1-|z|^2$ and hence $B_v = B_w$ be the classical Bloch space. Then $C_\phi : B_v \to B_v$ is compact if and only if*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0.$$

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